The evaluation of linear attenuation coefficients by computer assisted tomography.

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Ph.D. University of Edinburgh 1980

The work reported in this thesis was performed by the undersigned except where stated otherwise.

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Abstract

At the Department of Medical Physics, Edinburgh University, an interest was expressed in using C.A.T. scanning to obtain linear attenuation coefficients for use in Radiotherapy treatment planning. At the time (1971 - 1972) there were no commercially available whole body scanners and it was considered possible that one might be built by the department. This thesis reports some of the work done to access the feasibility of such a project.

A number of reconstruction algorithms extant at that time are examined and three shortcomings are revealed:-

1) there are no expressions for the errors on the output

2) the algorithms do not make proper allowance for the fact that the inverse Radon transform is discontinuous

and 3) the mathematical model used to represent the process of C.A.T. scanning does not allow for the effect of collimation or sampling.

A more detailed model of the scanning process is given and algorithms for its inversion are derived based on constrained optimisation.

Consideration of the errors inherent in the algorithm leads to a discussion of how to design a machine to achieve a given quality of picture.

To test the accuracy of the theory developed, an experimental machine was built and the thesis closes with a discussion of the results obtained from it.

Index

Chapter 1 : Introduction

Chapter 2: Reconstruction Theory

Chapter 3: Prediction Problems

Chapter 4: Equipment

Chapter 5: Software

Chapter 6: Computational Results

Chapter 7: Concluding discussion

Preface

.1 Motivation for work reported in thesis

At the radiotherapy centre in Edinburgh it was thought that the method of inhomogeniety correction used in the treatment planning software might be a major source of planning errors. It was therefore decided to try and remove these potential errors by introducing corrections based on measurements of the attenuation and/or absorption coefficients of the tissue in sections of each individual patient.

These corrections required two things: the development of a method for measuring the coefficients at different points in any typical patient section and the adaption (or complete rewriting) of the treatment planning software to make use of this new data. To develop a method for measuring the coefficients presupposes that one has decided on the details of what measurements are required, in particular what accuracy (density resolution) and spatial resolution are needed in the final answer. Unfortunately it was not possible to specify either of these parameters as there was no prior experience of treatment planning with this type of data on which to base such estimates and the opinions of those concerned with the project varied as to the likely values for the parameters. This one fact shaped the whole way in which the project developed.

When trying to develop a method (in this case a machine plus data processing software) to take particular measurements there are two ways in which one can proceed:-

EITHER One can just get on with building the machine from no more than an outline of the general principles on which it is to work. In this case there are likely to be many design decisions which are made more or less arbitrarily (e.g. angles, tolerances, etc.). When the building is completed one then uses the machine and assesses the quality of results that it produces. The machine design is then progressively revised until, hopefully, it produces the required result.

OR One can design the machine on paper only and in such a way that all the major design parameters are kept as unspecified variables. Following this one may further be able to assess, again on paper, the quality of results that the machine is capable of producing in terms of the major design parameters. One may then adjust (on paper) numerical values for the various machine parameters until one arrives at a set of values which represent the most easily built machine capable of producing the required quality of result.

The first approach requires that one knows just what final results are being aimed at otherwise it is impossible to decide whether the design meets the requirements or not. To a large extent the second approach frees one from this constraint as one is in effect designing not one machine but a whole family of machines. In fact it is only necessary to know the quality of final result required in order to perform the last stage of the design, and the design (though not necessarily the machine) can easily be changed subsequently if it is decided that some other quality of results is more appropriate. The danger of the second approach is that it is only necessary to make one mistake in the design calculations and it may subsequently be discovered that a great deal of time, effort and money has been used in building the wrong machine.

Because it was not possible to specify in advance the density and spatial resolution required, the second approach to the machine design was adopted. It was further decided that a small experimental machine would be built in order to try and check the accuracy of the design procedure and data processing.

Two physical principles were considered as possible bases for taking such measurements:-

1) an adaption of the Compton scattering approach suggested by Farmer (Farmer et al, 1971).

2) the computerised tomography (C.T.) approach (see §1·2 and §1·4). After brief consideration of the relative merits of the two methods it was decided to proceed with the C.T. approach.

Finally, it was necessary to decide between the use of an X-ray tube or an isotope as the radiation source. The isotope source was chosen for four reasons:-

a) it was felt on the basis of the calculation given in §4·2 that some of the larger commercially available sources were of sufficient specific activity.

b) the use of a mono-energetic source gives rise to a well defined mathematical model (see §2·1).

c) such a source was already available in the department.

d) the use of an isotope simplifies the construction of the experimental machine.

·2 Work covered by thesis

The previous section outlined briefly the motivation for this research project together with four major factors which affected its direction of development namely that:-

a) it was decided to tackle first the measurement of attenuation and/or absorption coefficients rather than the rewriting of the planning software.

b) a more theoretical approach to the scanner design was adopted due to the impossibility of specifying the resolution required.

c) it was decided to use the reconstruction method to obtain the results.

d) it was decided to use an isotope source.

For the present purposes these factors are not discussed further but simply taken as axiomatic and this thesis reports some of the work done in attempting to solve the problem:-

"of developing and testing

1) the necessary data processing software and

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for a machine capable of measuring linear attenuation coefficients to an arbitrary spatial and density resolution, under the assumptions that the machine is to work on the C.T. principle and to use an isotope source of radiation."

In practice this work falls naturally into six parts:-

α) development of a mathematical model to represent the scanning procedure

β) development of an inversion procedure for the mathematical model

γ) the reduction of the inversion procedure to an algorithm suitable for computer implementation

δ) development of the relations between the parameters representing scanner design and those representing output picture quality

ε) the reduction of scanner design/picture quality relations to an algorithm suitable for predicting the main scanner design parameters in terms of the picture quality parameters

ζ) the building of a small experimental scanner to provide some verification of the mathematical results.

The degree of success which attended these various endeavours will be seen in the following chapters and will be summarised in the final chapter.

·3 Background required of reader

There are two factors which have caused problems during the course of the research work reported in this thesis - the explosion of information on C.T. scanning since 1970 and the wide range of disciplines involved in C.T. scanning.

Dr Gordon (Gordon, 1976) has produced a bibliography of literature relevant to C.T. scanning which contains references to about 3,500 papers most of which have been published in the period 1970 - 1976. It is clear that, within the space constraints of a thesis, there can be no question of giving a comprehensive survey of this many papers and all that is attempted is to give a brief account in §1·3 and §1·4 of those papers of most immediate relevance to the present work while leaving many significant papers unmentioned. It is therefore necessary that the reader should already have at his finger tips a good background knowledge of the literature and be thoroughly familiar with the concepts of C.T. scanning.

The following disciplines have been involved in the work reported here:-

Computer programming
X-ray physics
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analogue
nucleonics
Digital image processing
Statistics
Mechanical engineering
Mathematics : functional analysis
real and complex analysis
Fourier analysis
matrix theory
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non-linear programming
generalised function theory

The reader will find that he can get by without most of these if he is prepared to take various results on trust, however the four subjects

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The reader who has got this far will have realised that there is a good deal of mathematics in the thesis and a word or two about the way in which it is used seems to be in order. It is the author's experience that all too often scientists tend to divide themselves up into theoretical men and practical men. Readers who put themselves too firmly in either group are likely to be disappointed with what they find here. The mathematician who hopes for a formal treatise on the mathematics of C.T. scanning will find that only in one or two appendices is the theorem and proof approach adopted while the main chapters are more of a discussion containing a number of 'reasonable' assumptions. Likewise the string and sealing wax man will find that there are more integral signs in the text than is healthy. The approach used in the thesis is a middle of the road one and most of the time the mathematics is used not as an end in its own right but simply as the most convenient language in which to discuss essentially physical ideas and it must therefore be read accordingly.

In summary, the reader will find it essential to have a knowledge of :-

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the literature on reconstruction theory
real and complex analysis
Fourier analysis
matrix theory
digital image processing

and in addition be willing to use mathematical notation as a convenient language for discussing physical situations.

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The author is uncomfortably aware of a number of loose ends, for example:-

1) The choice of $f \in \mathcal{L}^2(B^2)$ is not really suitable for some of the discussion in §1·3 and §1·4.
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4) Given the mathematics, the scanner and the software much useful practical work could have been done in addition to that reported in chapter 6.
5) In chapter 2 regularisation techniques are applied to the inversion of the Radon transform. It is possible to develop

a parallel theory of regularised parametric inversion methods. The mathematics for this has been done but it was not reduced to computer programmes and is completely omitted from this thesis.

Such loose ends resulted mainly from a lack of time due to the fact that the project turned out to be more time consuming and more complex than was originally envisaged and it is the author's hope that their existence will be treated with sympathetic tolerance by the reader.

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ILLUSTRATIONS

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Chapter 1 : Introduction

·1 Introduction

One of the major problems in using reconstruction theory for C.T. scanning is the inversion of the Radon transform and this chapter is devoted to discussing this transform and some of the relevant published literature about it.

The chapter begins by bringing together two sources of information. The first is a summary of the relevant properties of the Radon transform (a formal exposition can be found in §1·1·4) and the second is a selection of the more relevant published literature. This literature is then discussed in the light of the transform's properties with emphasis on the discontinuity of the inverse transform.

Finally, arising from the aims outlined in the preface, a list of requirements is given for the reconstruction theory and the literature is again examined to see how well it meets these requirements.

It is emphasised once again that, as intimated in §·3 of the preface, the discussion of the literature in this chapter is not intended as an introductory survey but as discussion of the overall features of a body of literature with which the reader is assumed to be already familiar. Those who would like an introduction to the literature could read Brooks and di Chiro (Brooks et al, 1976) or I.E.E.E. (I.E.E.E. NS-21, 1974).

·2 Physical principles of reconstruction method

Denote by f the linear attenuation coefficient at any point in a transverse section of a patient. The transmission of γ-radiation along a narrow straight line L through the section (see F·1) gives rise to the equation

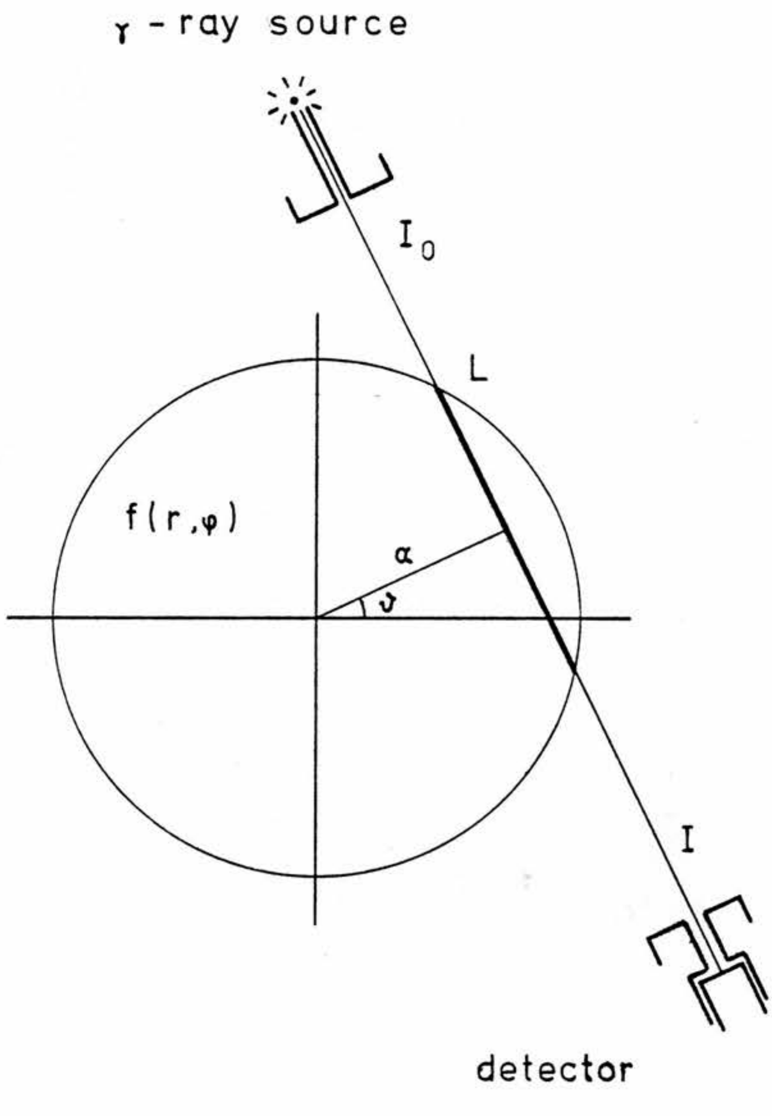


fig. 1·1 : Transmission scan

$$I = I_0 \exp\left\{-\int_L f\right\}$$

where I_0 is the incident radiation intensity

I is the final intensity

and $\int_L$ denotes a line integral along L

If the position of L is varied and is specified as $L(\alpha,\theta)$ where α and θ are as shown in F·1 then a new function $p(\alpha,\theta)$ can be defined by

$$p(\alpha,\theta) = \int_{L(\alpha,\theta)} f \qquad \cdot 10$$

The idea of estimating f by the reconstruction method consists of four stages:-

1) measure I_0 (which is assumed to be constant)
2) measure $I(\alpha,\theta)$ for a number of sample points (i.e. various α and θ)
3) using the values of I_0 and $I(\alpha,\theta)$ estimate $p(\alpha,\theta)$ at the same sample points from the relation

$$p(\alpha,\theta) = \ln(I_0/I(\alpha,\theta)). \qquad \cdot 11$$

4) estimate f from the calculated values of p by using an inverse to the relation E·10.

Clearly there is no problem in estimating p from I_0 and I, but the method of estimating f from p is more difficult and is now examined in more detail.

·3 The Radon transform

The mapping $f \to p$ given in E·10 is usually known as the Radon transform which is here denoted by $\mathcal{R}$ so that E·10 may be written $p = \mathcal{R} f$.

In this notation, stage (4) of the reconstruction method requires that one should find some process which evaluates $f = \mathcal{R}^{-1}p$. It is therefore natural to consider the properties of $\mathcal{R}$ and in particular whether $\mathcal{R}^{-1}$ exists.

·31 Definition of $\mathcal{R}$

In defining $\mathcal{R}$ it is necessary to consider the properties of f. Since f denotes the linear attenuation coefficient at different points, it will be finite and positive (or at least non-negative) within the section and, for practical purposes, zero outside the section. In addition, any patient section is of finite extent thus for the coordinate system given in F·1 there is a number D (the diameter of the section) such that the patient section is entirely contained in the area $|r| < D/2$. The function f may therefore be taken to be a bounded non-negative function defined on $\mathbb{R}^2$ with support in $\{|r| \leq D/2\}$ for some D.

The next question to consider is the continuity of f. A chest section, for example, will contain both muscle/bone and muscle/lung interfaces and it is therefore suspect to consider f to be adequately represented by a continuous function.

Combining these facts, $\mathcal{R}$ is thought of as being defined on the Lebesgue square integrable functions whose support is in $\{|r| \leq D/2\}$. Using this domain for $\mathcal{R}$ ensures that sufficient functions are included to represent the various cross-sections which may be encountered in practice. A formal definition of $\mathcal{R}$ is given in §1·1·43 where it is assumed without loss of generality that the support of f is contained in the unit circle.

·32 Properties of $\mathcal{R}$

From the definition of $\mathcal{R}$ indicated above and given formally in §1·1·43 it is possible to prove three theorems which are summarised and discussed in this section. Once again the formal proofs etc. are given in §1·1·4 where it is assumed without loss of generality that the support of f is contained in the unit circle.

·321 $\mathcal{R}^{-1}$ and the Projection theorem

If (r,ϕ) and (R,Φ) are the polar coordinates of points in $\mathbb{R}^2$ then, by definition, the two dimensional Fourier transform (F.T.) of $f(r,\phi)$ is

$$F(R,\Phi) = \int_0^{2\pi}\int_0^{\infty} f(r,\phi)e^{-2\pi i r R\cos(\phi-\Phi)}r\,dr\,d\phi$$

(see §1·1·12) and from the Fourier integral theorem

$$f(r,\phi) = \int_0^{2\pi}\int_0^{\infty} F(R,\Phi)e^{2\pi i R r\cos(\Phi-\phi)}R\,dR\,d\Phi\ . \qquad \cdot 12$$

Similarly given $p(\alpha,\theta)$ defined as in E·10, then by definition the one dimensional Fourier transform of p with respect to the first variable is

$$P(R,\Phi) = \int_{-\infty}^{\infty} p(\alpha,\Phi)e^{-2\pi i\alpha R}\,d\alpha. \qquad \cdot 13$$

The projection theorem states that

$$P(R,\Phi) = F(R,\Phi) \qquad \cdot 14$$

(see 1·1·44). Note that this theorem both implies the existence of $\mathcal{R}^{-1}$ and gives an explicit formula for it, for if $p(\alpha,\theta)$ is given P can be calculated from E·13, hence from E·14 F is known and f may then be calculated from E·12. If the one and two dimensional F.T. are denoted by $\mathcal{F}_1$ and $\mathcal{F}_2$ then E·12, E·13 and E·14 may be summarised as

$$f = \mathcal{F}_2^{-1}\mathcal{F}_1 p\ . \qquad \cdot 15$$

This result is the basis of all the non-parametric Fourier inversion techniques although for numerical work it is usually restated in one of the two forms given below.

·322 The Back Projection theorem

Equation ·12 may be rewritten

$$f(r,\phi) = \frac{1}{2} \int_0^{2\pi}\int_{-\infty}^{\infty} F(R,\Phi)e^{2\pi iRr\cos(\Phi-\phi)}|R|\,dR\,d\Phi \qquad \cdot 16$$

and the inner integral

$$\int_{-\infty}^{\infty} |R|\ F(R,\Phi)e^{2\pi iR\alpha}dR \qquad \cdot 17$$

where $\alpha = r\cos(\Phi-\phi)$, is the F.T. of the product of two functions. Denote the F.T. of $|R|$ by $s_1(\alpha)$ (see below), from E·14 and the inverse of E·13 the F.T. of $F(R,\Phi)$ is $p(\alpha,\Phi)$ thus E·17 may be written

$$s_1(\alpha) \ *_1\ p(\alpha,\Phi)$$

where $*_1$ denotes convolution with respect to the first variable in p. Substituting this in E·16 gives

$$f(r,\phi) = \frac{1}{2}\int_0^{2\pi} d\Phi\ \{s_1(\alpha) *_1 p(\alpha,\Phi)\}\Big|_{\alpha = r\cos(\Phi-\phi)}$$

$$= \int_0^{\pi} d\Phi \int_{-\infty}^{\infty} d\alpha'\ s_1(r\cos(\Phi-\phi) - \alpha')p(\alpha',\Phi)$$

Cormack (Cormack, 1973 : p202 E16) gives an expression closely related to this and has also pointed out that it was first proved by Radon (Radon, 1917). More recent proofs are given in Berry and Gibbs (Berry et al, 1970) and John (John, 1955).

These results may be conveniently summarised in operator notation. Given a function $t(\alpha,\theta)$ denote by $\mathcal{B}t$ the function

$$(\mathcal{B}t)(r,\phi) = \frac{1}{2}\int_0^{2\pi} t(r\cos(\theta-\phi),\theta)d\theta$$

By putting $t(\alpha,\Phi)$ equal to the expression E·17 it is straightforward to see that E·16 may be written

$$f = \mathcal{B}\,\mathcal{F}_1^{-1}(|R|\,\mathcal{F}_1 p) \qquad \cdot 18$$

and the manipulation that follows is simply

$$f = \mathcal{B}\{\mathcal{F}_1^{-1}(|R|) *_1 p\}$$

$$= \mathcal{B}\{s_1(\alpha) *_1 p\} \qquad \cdot 19$$

Either of E·18 or E·19 is known as the Back Projection theorem (some authors refer to them by the name 'Filtered Back Projection') and while they are mathematically equivalent (and also equivalent to E·15) there are significant computational differences which are discussed in §5. Formal proof of E·18 may be found in §1·1·46.

(N.B. The function $s_1(\alpha)$ has been left unidentified because it exists only as a generalised function. Since distribution theory is not required elsewhere in the thesis it seems disproportionate to introduce it just for this section.)

·323 Continuity of $\mathcal{R}^{-1}$

In this section the validity of an assumption often made in the published literature on inversion of the Radon transform is examined.

·3231 Demonstration of assumption

·32311 Fourier based techniques

Consider the evaluation of f by the use of any one of the equations E·15, E·18 or E·19. If f denotes the true distribution of attenuation coefficients and p its Radon transform then

$$f = \mathcal{R}^{-1} p$$

and given p one can evaluate f (at least in principle). However in the practical situation one knows only an approximate estimate for p. Denote this estimate by $\hat{p}$ and its error by ε so that

$$\hat{p} = p + \varepsilon .$$

From this data the best that can be achieved is to arrive at an estimate $\hat{f}$ of f and it is natural to calculate $\hat{f}$ using

$$\hat{f} = \mathcal{R}^{-1}\hat{p} ,$$

the question is 'Will $\hat{f}$ be a good estimate of f?'. Consider the difference between them;

$$\begin{aligned}\hat{f} - f &= \mathcal{R}^{-1}(p + \varepsilon) - \mathcal{R}^{-1}p \\ &= \mathcal{R}^{-1}\varepsilon\end{aligned}$$

and to be sure that $\hat{f}$ is a good estimate of f it is required that

$$\varepsilon \text{ small} \Rightarrow \mathcal{R}^{-1}\varepsilon \text{ small}$$

or equivalently

$$\varepsilon \to 0 \quad \Rightarrow \mathcal{R}^{-1}\varepsilon \to 0$$

and this is the same as requiring that $\mathcal{R}^{-1}$ is continuous.

·32312 Iterative techniques

Once again denote the distribution of attenuation coefficients by f and its Radon transform by p. The iterative techniques for calculating f from p are based on the idea of generating a sequence of estimates $\hat{f}_n$ such that $\hat{f}_n \to f$. This is done as follows: given $\hat{f}_n$ then define $\hat{p}_n$ by

$$\hat{p}_n = \mathcal{R}\hat{f}_n$$

the next 'improved' estimate of f is then given by

$$\hat{f}_{n+1} = \hat{f}_n + \mathcal{B}(p - \hat{p}_n)$$

where $\mathcal{B}$ is an operator which generates a correction for $\hat{f}_n$ from a knowledge of $p - p_n$. In practice the correction operator differs from one iterative technique to another, but it is always chosen so that $\hat{p}_n \to p$. In this way two sequences $\{\hat{p}_n\}$ and $\{\hat{f}_n\}$ are generated and $\mathcal{B}$ is chosen so that $\hat{p}_n \to p$. This raises the question 'Is the condition $\hat{p}_n \to p$ sufficient to ensure that $\hat{f}_n \to f$?' If ε_n is defined by $\varepsilon_n = \hat{p}_n - p$ then $\hat{f}_n - f = \mathcal{R}^{-1}\varepsilon_n$ and in order to ensure that $\hat{f}_n$ converges to f it is required that

$$\varepsilon_n \to 0 \Rightarrow \mathcal{R}^{-1}\varepsilon_n \to 0$$

i.e. that $\mathcal{R}^{-1}$ is continuous.

In fact the situation is somewhat worse than this for, as was noted above, one only knows $\hat{p} = p + \varepsilon$ and not p so that a $\{\hat{p}_n\}$ is generated such that $\hat{p}_n \to \hat{p}$ or equivalently $\varepsilon_n \to \varepsilon$.

·3232 Validity of assumption

The way in which many authors, whether interested in Fourier or Iterative techniques, assume continuity of $\mathcal{R}^{-1}$ has been indicated and the problem is that this assumption is not true. In §1·1·48 it is proved by counter example that $\mathcal{R}^{-1}$ is discontinuous so that, given an estimate $\hat{f}$ of f, the fact that $\hat{p} = \mathcal{R}\hat{f}$ is close to $p = \mathcal{R}f$ is no guarantee that $\hat{f}$ is close to f.

·4 Extant methods of inverting $\mathcal{R}$

Those fall into two groups : those based on the Fourier analysis approach indicated in §·321 and §·322 and widely known as 'Fourier techniques' and the 'Iterative techniques' based on the ideas outlined in §·32312. The Fourier techniques can be further classified into non-parametric and parametric methods.

·41 Non-parametric Fourier techniques

Because of the finite size of the detector used to obtain the data $I(\alpha,\theta)$ and because $I(\alpha,\theta)$ is measured only at certain sample points then the high frequency information in $p(\alpha,\theta)$ (regarded as a function of α only) will be attenuated. Assuming this high frequency filtering can be represented by rectangular filtering in Fourier space, then the data measured is not $p(\alpha,\theta)$ as desired but

$$p(\alpha,\theta) \ *_1 \mathcal{F}_1^{-1}\left\{\Pi\left(\frac{R}{2\omega_c}\right)\right\}$$

where ω_c is the high frequency cut-off. Using $\hat{f}$ to denote the value of f which can be reconstructed from the degraded data, it follows from E·18 that

$$\hat{f} = \mathcal{B}\,\mathcal{F}_1^{-1}\left\{|R|\,\Pi\left(\frac{R}{2\omega_c}\right)\,\mathcal{F}_1 p\right\}.$$

Define $s_2(\alpha,\omega_c)$ by

$$\begin{aligned} s_2(\alpha,\omega_c) &= \mathcal{F}_1^{-1}\left\{|R|\,\Pi\left(\frac{R}{2\omega_c}\right)\right\} \\ &= \mathcal{F}_1^{-1}\left\{\omega_c\,\Pi\left(\frac{R}{2\omega_c}\right) - \omega_c\,\Lambda\left(\frac{R}{\omega_c}\right)\right\} \\ &= 2\omega_c^2 \,\mathrm{sinc}\,(2\omega_c\alpha) - \omega_c^2\,\mathrm{sinc}^2(\omega_c\alpha) \end{aligned}$$

then

$$\hat{f}(r,\phi) = \frac{1}{2}\int_0^{2\pi} d\theta\,\left\{p(\alpha,\theta) \ *_1 s_2(\alpha,\omega_c)\right\}\Big|_{\alpha = r\cos(\theta-\phi)} \qquad \cdot 21$$

This is the convolution method suggested by Bracewell (Bracewell et al, 1967). (N.B. There are a number of equations in this article in which the factor M, here denoted by ω_c, has been omitted) see also (Ramachanchan et al, 1971).

Gilbert (Gilbert, 1972) has examined the errors introduced by angular sampling which must inevitably take place if E·21 is used in practice. In this case it will often happen that $p(\alpha,\theta)$ is known only for $\theta = \theta_j$

where $\theta_j = \pi(j-1)/M$ and $j = 1(1)M$ and E·21 may be approximated by

$$\hat{f}(r,\phi) = \frac{\pi}{M} \sum_{j=1}^{M} p(\alpha, \theta_j) *_1 s_2(\alpha,\omega_c) \Big|_{\alpha = r\cos(\theta_j - \phi)}$$

Manipulation of this formula shows that

$$\hat{f}(r,\phi) = f(r,\phi) *_2 2 \int_0^{\omega_c} R \sum_{n=-\infty}^{\infty} J_{2nM}(2\pi rR) e^{i2nM(\phi + \frac{\pi}{2})} dR \qquad \cdot 211$$

where $*_2$ denotes convolution over $\mathbb{R}^2$. (No proof of this result is given here. It has been proved incorrectly in (Gilbert, 1972) whose E9 is misquoted from (Waser, 1955) E7.) The same paper also examines in some detail the effect of convolution with the integral term.

However, in practice E·21 is undesirable for use in C.T. because of the sharp cut-off of the filter $s_2(\alpha,\omega_c)$. While it is quite feasible to introduce such sharp cut-offs when performing digital filtering experience in image processing shows that it is best avoided as the sharp cut-off in Fourier space causes ringing near discontinuities in real space. This problem has been examined specifically in relation to C.T. by Chesler (Chesler et al, 1975) who suggests that instead of combining the original filter $s_1(\alpha)$ with a rectangular window (giving $s_2(\alpha,\omega_c)$) one should combine it with a Hanning window giving a filter $s_3(\alpha)$ where

$$s_3(\alpha) = \mathcal{F}_1^{-1}\left\{|R|\left(1 + \cos\frac{\pi R}{2\omega_c}\right)\right\}$$

and from this it may be shown by the use of standard techniques that

$$s_3(\alpha) = \frac{1}{2} s_2(\alpha - \frac{1}{4\omega_c}, 2\omega_c) + s_2(\alpha, 2\omega_c) + \frac{1}{2} s_2(\alpha + \frac{1}{4\omega_c}, 2\omega_c).$$

The author is not aware of any papers which derive a result parallel to that of E·211 but covering the filter s_3 rather than s_2.

·42 Parametric Fourier techniques

To avoid the various numerical problems inherent in E·15, E·18 or E·19 many authors (Crowther et al, 1972) (Herlitz, 1963) (Klug et al, 1972)

(Maldonado et al, 1966) (Marr, 1974) (Smith et al, 1973) (Zeither, 1974) suggest expanding the functions f and p in terms of a set of basis functions.

Suppose $\{\psi_j(r,\phi)\}$ is a set of functions which is complete and orthogonal (with respect to some weight function) over the disc $\{|r| \le D/2\}$ and zero outside. Then f may be expanded as

$$f(r,\phi) = \sum_j a_j \psi_j(r,\phi) \qquad \cdot 22$$

whence

$$p = \mathcal{R}f = \sum_j a_j \mathcal{R}\psi_j \qquad \cdot 23$$

If the functions $\mathcal{R}\psi_j$ are evaluated (preferably analytically but possibly numerically) then, given p, the constants a_j can be found from E·23. By putting these a_j into E·22 values of f can be obtained.

Two sets of basis functions have received wide-spread attention:-

1) Crowther and Klug (Crowther et al, 1972) (Klug et al, 1972) have suggested using the functions

$$\psi_{ns}(r,\phi) = \begin{cases} J_n(2\lambda_{ns} r/D)e^{in\phi} & |r| \le D/2 \\ 0 & |r| > D/2 \end{cases}$$

where $r = -\infty(1)\infty$, $s = 1(1)\infty$ and λ_{ns} denotes the s-th zero of J_n. In this case $\mathcal{R}\psi_{ns}$ does not appear to have any convenient analytic form (but can be evaluated numerically), it does however have various interesting sampling properties.

2) Many authors (Herlitz, 1963) (Maldomado et al, 1966) (Marr, 1974) (Smith et al, 1973) (Zeitler, 1974), apparently independently, have suggested using

$$\psi_{ns}(r,\phi) = \begin{cases} R^n_{n+2s}(2r/D)\, e^{in\phi} & |r| \le D/2 \\ 0 & |r| < D/2 \end{cases}$$

where $n = -\infty(1)\infty$, $s = 0(1)\infty$, R^n_{n+2s} is the Zernike polynomial defined by

$$R^{n}_{n+2s}(\rho) = (-1)^{s} \rho^{|n|} P_{s}^{(|n|,0)}(1 - 2\rho^{2})$$

with P as the Jacobi polynomial. Properties of R^{n}_{n+2s} can be found in Born and Wolf (Born et al, 1955). With these ψ_{ns} it may be shown that

$$\mathcal{R}\psi_{ns}(\alpha,\theta) = \begin{cases} \frac{D}{|n| + 2s+1}\sqrt{1 - \left(\frac{2\alpha}{D}\right)^{2}}\; U_{|n|+2s}\left(\frac{2\alpha}{D}\right) e^{in\theta} & |\alpha| \le D/2 \\ 0 & |\alpha| > D/2 \end{cases}$$

where U_j is the Chebyshev polynomial of the second kind defined by

$$U_j(\cos \phi) = \sin((j + 1)\phi)/\sin \phi.$$

With these functions it may be shown that the ψ_{ns} are orthogonal with the unit weight function and the $\mathcal{R}\psi_{ns}$ (considered to be defined in $[-D/2, D/2] \times [0,2\pi]$) are orthogonal with weight function $\{1 - (2\alpha/D)^{2}\}^{-\frac{1}{2}}$. In addition the $\psi_{ns}(r,\phi)$ when expressed in Cartesian coordinates, are polynomials in x and y (Marr, 1974). If the parametric inversion technique is to be used then these properties make this particular set of basis functions very attractive.

·43 Iterative techniques

There are a number of iterative algorithms available all of which are based on the concepts of §·32312 the differences being in the choice of the operator $\mathcal{C}$, for example ART - (Gordon et al, 1970) (Bender et al, 1970) (Herman et al, 1973a), SIRT - (Gilbert, 1972), EFIRT - (Crowther et al, 1974). All these are statements of an 'intuitively reasonable' idea and, unlike the Fourier techniques, they have not been built upon a carefully laid mathematical basis.

Comparative studies of the effectiveness of the methods have been made (Herman, 1972) (Herman et al, 1973b), however these are all based on the reconstruction of a few trial objects and "... theoretical comparisons are made difficult because there is no common mathematical foundation for the reconstruction algorithms ..." (Gordon et al, 1973). None of these methods give a priori estimates of the resolution (density or spatial) in the final reconstruction.

·5 Inversion methods and continuity

In §·323 it was seen that $\mathcal{R}^{-1}$ was discontinuous and the significance of this for both the Fourier and Iterative techniques was illustrated. In view of the fundamental nature of this result it is perhaps surprising that any of the reconstruction techniques function at all. In this section the discontinuity of $\mathcal{R}^{-1}$ is examined first in relation to the basic theorems of §·32 and then in relation to the various algorithms outlined in §·4 in order to understand why some of them work in practice despite the instability of $\mathcal{R}^{-1}$.

·51 Discontinuity of $\mathcal{R}^{-1}$ and the reconstruction theorems

In §·321 and §·322 three theorems were stated

$$f = \mathcal{F}_2^{-1}\mathcal{F}_1 p$$

$$f = \mathcal{B}\mathcal{F}_1^{-1}\{|R|\mathcal{F}_1 p\}$$

$$f = \mathcal{B}\{ s_1(\alpha) *_1 p\}$$

and each of these has been used by various authors as the basis for reconstruction algorithms. We now pause to examine how the discontinuity of $\mathcal{R}^{-1}$ is manifest in the different theorems.

In the first equation the discontinuity of $\mathcal{R}^{-1}$ is subtly hidden. In the formal presentation of the properties of $\mathcal{R}$ this equation is more accurately stated as $\mathcal{R}^{-1} = \mathcal{F}_2^{-1}\mathcal{P}^{-1}\mathcal{F}_1$ (see §1·1·45) and it is the apparently trivial operator $\mathcal{P}$ whose inverse is discontinuous. However, this is not taken further as this equation is only rarely used for numerical work.

In the second equation the discontinuity appears in the process of multiplying $\mathcal{F}_1 p$ by $|R|$. It will be immediately obvious to anyone with image processing experience that changing from p to $\mathcal{F}_1^{-1}\{|R|\mathcal{F}_1 p\}$ will enhance any noise present in p, and that the operation will be unstable.

In the third equation, this problem appears in calculating the convolution $s_1(\alpha) *_1 p$ as the function $s_1(\alpha)$ and the integral do not exist in the classical sense and must really be considered in the context of generalized function theory.

·52 Discontinuity of $\mathcal{R}^{-1}$ and the reconstruction algorithms

·521 Non-parametric Fourier techniques

The original theorem on which most of these are based is

$$f = \mathcal{B}\mathcal{F}_1^{-1}\{|R|\ \mathcal{F}_1 p\}$$

and the suggestion (in the case of Bracewell's method) is that since p is band limited by the data collection method then

$$(\mathcal{F}_1 p)(R,\theta) = \Pi\left(\frac{R}{2\omega_c}\right)(\mathcal{F}_1 p)(R,\theta)$$

and instead of filtering $\mathcal{F}_1 p$ by $|R|$ one can filter it by $|R|\ \Pi(R/2\omega_c)$ with the same result. However, this change, which appears in the guise of a mathematical convenience to allow one to pass from convolution with $s_1(\alpha)$ to convolution with $s_2(\alpha,\omega_c)$ (see §·41), is in fact a necessity. This is because the instability of multiplying by $|R|$ arises from the excessive enhancement of small high frequency noise components and the change to $|R|\ \Pi(R/2\omega_c)$ cures this (if ω_c is chosen correctly (see §·622)).

·522 Parametric Fourier techniques

In the practical application of the parametric methods, $p(\alpha,\theta)$

is only known for a finite number of sample points (in both α and θ) and it is therefore only possible to fit a finite number of basis functions to the available data. In practice the basis functions corresponding to lower frequencies are always fitted first (i.e. low order Bessel function, exponentials, polynomials etc.) and so a rather ill defined form of band limiting is inherent.

·523 Iterative techniques

Unlike the Fourier techniques most of the iterative techniques do not contain approximations which increase their stability. It is hardly surprising that in practice many of them diverge if the number of iterations is too large.

·6 Discussion

The discussion of this chapter has covered some of the more important literature on the theoretical aspects of C.T. scanning, we now consider how the ideas in the published literature match up with the aims of this project as outlined in §·2 of the Preface. This comparison centres around the theory of reconstruction and the associated software because little has been published on the hardware.

·61 Required characteristics of theoretical work

The eventual aim of the project is to arrive at a body of software and a hardware design procedure, this requires that one first develop the theory of which these are the practical fruit. It is desirable that the theory should have the following characteristics.

·611 Model

The theory must contain a theoretical model representing both the actual scanning process and all the significant design decisions which will be made when the scanner is built.

·612 Resolution

The theory must allow an a priori estimate of the density and spatial resolution that can be expected in pictures produced by the scanner.

·613 Optimal data processing

It is desirable that the theory should show how a given set of data may be processed in order to obtain the best possible picture.

·614 Optimal scanner design

Given a stated output picture quality it is desirable that the theory should show how the scanner may be built in order to obtain such a picture in the most efficient way.

·62 Characteristics of extant theoretical work

The extent to which the current literature (outlined above) exhibits the four attributes listed in §·61 is now discussed.

·621 Model

·6211 Fourier methods

The standard mathematical model used in the extant literature on Fourier methods is that given in §·2. Note its chief features, it assumes that all the functions are defined and measured on a continuum, that there is no noise inherent in the data and that any effects due to collimation, sampling, etc. may be neglected. It is customary to derive expressions for $\mathcal{R}^{-1}$ on this basis and then to make alterations to the expression to allow for the fact that the data is sampled, degraded by statistical noise, etc. As a result of these omissions in the model, the theory to which it gives rise only present a rather hazy guide, if any at all, on such matters as how to choose the collimator size, or how to match the filter cut-off frequency to the data (see §·622).

·6212 Iterative methods

The usual model used for the iterative methods is to assume that f is made up of a square matrix of cells on each of which f is constant, and the data is assumed to be the projections of such a function with p exactly known at certain sample points so that no allowance is made for collimation or data statistics. The dangers of this model have been eloquently underlined by the variety of claims (often contradictory) which have been made for these techniques.

It is also interesting to note just how strong is the assumption that a genuine f may be approximated by such a step function model. Smith (Smith et al, 1973 : Appendix 1) shows that with an object of this step function form it is possible to reconstruct the object from a single projection if there is no noise present.

·622 Resolution

·6221 Fourier methods

Consider first the concepts of density and spatial resolution.

Suppose it were possible to measure p at all points (i.e. on a continuum) and that ideal data processing could be used so that the only source of image degradation being considered is due to the fact that one measures $\hat{p} = p + \varepsilon$ rather than p. In this case the reconstructed image will be $f + \mathcal{R}^{-1}\varepsilon$ and the correct image f will be obscured by the high frequency noise $\mathcal{R}^{-1}\varepsilon$. The density resolution is a measure of how small a change in the value of f can be seen in the reconstructed section despite the additional noise, and one can expect to see such changes down to the point at which the change becomes comparable with the standard deviation of the noise $\mathcal{R}^{-1}\varepsilon$, beyond this one will not know whether a change in $f + \mathcal{R}^{-1}\varepsilon$ is arising from a genuine change in f or arising from the addition of $\mathcal{R}^{-1}\varepsilon$.

It has already been illustrated in §·322 and §·4 that whereas $\mathcal{R}^{-1}$ may be correctly implemented as $\mathcal{B}\{s_1(\alpha) *_1 p\}$, in practice it is implemented with some other filter (e.g. as $\mathcal{B}\{s_3(\alpha) *_1 p\}$). This change of filters from s_1 to some alternative always takes the form of adding a low pass filter, and (even in the absence of noise on the data) this will impose its own limit to the amount of fine spatial detail that can be seen. The spatial resolution is a measure of this limit.

Various authors have given expressions for the density and spatial resolution obtainable from a given set of data. In the case of spatial resolution these are usually based on either the approach of Bracewell (Bracewell et al, 1967) (as in §3·313 E·35 which leads to $\omega_c \le M/\pi D$) or on the approach of Gilbert (Gilbert, 1972) (Brooks et al, 1978) (where consideration of E·211 leads to the condition that $\omega_c \le 2M/\pi D$, though this condition appears to be necessary rather than sufficient). In the case of density resolution the derivation of the expressions usually contain non-trivial restrictions (e.g. giving an expression which is only valid at the centre of a uniform circular object, or assuming that all data observations have the same variance) (see the references given in §3·52). However, leaving these point aside, there is a more fundamental criticism of the resolution expressions which is now discussed.

In the above discussion, the density and spatial resolution were introduced as two independent quantities; the density resolution being introduced with the assumption that the spatial resolution was unlimited and vice versa. This approach is typical of the literature known to the author, but it is open to question. In §·41 filters s_2 and s_3 were introduced; both give rise to a spatial resolution determined by ω_c but nothing was said about how to choose ω_c. Loosely speaking the spectrum of p will have the following character : its low frequency components will be large and error free, its intermediate frequency components will be smaller and a combination of genuine data and statistical noise, while its high frequency components will be entirely made up of statistical noise. Because of this the effect of choosing ω_c will be as follows : if ω_c is chosen too large useful information in the picture will be obscured by statistical noise, if ω_c is chosen too small, the noise will be reduced but useful information will also be removed unnecessarily. Because of this ω_c should be chosen to lie in the intermediate frequencies but which particular frequencies there are will vary depending on both the statistical noise present and on the particular p being examined. For this reason, ω_c , although basically determining the spatial resolution, should be chosen according to the data statistics and not just according to the sampling interval as is usually the case.

The density and spatial resolution are therefore intimately related and the attempt to define either independently of the other suggests that one may be about to choose ω_c independent of the data statistics. This in turn suggests that data processing is envisaged which may be very poorly suited to the data in question.

·6222 Iterative methods

The author is unaware of any expressions for density or spatial resolution which are applicable to the iterative techniques.

·623 Optimal data processing

As seen in ·6221 the cut-off frequency used in filtering p needs to be matched to the data being processed in order to obtain the maximum amounts of information in the picture. In the literature of which the author is aware, this is not done.

·624 Optimal scanner design

When building a scanner there is a wide range of choice as to what data is collected, the price for good data being an increased radiation dose to the patient. Since this is undesirable one therefore requires guidance on how the scanner should be built in order to obtain the required picture without radiating the patient unnecessarily. The author is unaware of any such design guidance in the literature.

·63 Conclusions

·631 Model

In this chapter three approaches to reconstruction have been examined and it has been seen that the mathematical models used all lacked sufficient detail, in particular no allowance was made for such things as collimation of the γ-ray beam and statistical noise in the data. Thus the first task is to develop a more detailed model and it will then be found that the inversion of $\mathcal{R}$ is only one part of the total inversion procedure.

·632 Reconstruction method

Having decided that a fuller inversion method is to be developed and that part of this inversion will be implementing $\mathcal{R}^{-1}$, there is still the question of which (if any) of the three techniques to use for inverting $\mathcal{R}$. For this there are two deciding factors:-

1) the requirements for an a priori estimate of the resolution (§·612) and optimal data processing (§·613) and scanner design (§·614) all require a sound mathematical basis for the reconstruction. Such a basis is extant

for the Fourier methods but not the iterative ones.

2) such timings as exist (and are known to the author) (e.g. Smith et al : 1973) suggest that the Fourier methods require less computer time.

Both these factors are in favour of the Fourier approach and this is therefore chosen. Finally, having decided on a Fourier method there is a choice of parametric or non-parametric and once again the factor of computer time is relevant and is in favour of the non-parametric method as this avoids the necessity for two dimensional curve fitting.

·7 Postscript

This survey of the literature was originally carried out at the start of the research project in order to make various decisions on the best way to develop the project, and, not surprisingly, while the author has been working away, so have several other people. The survey has been partially updated and in general is thought to be a fair assessment of the current state of the relevant literature, but there is one notable omission which should be mentioned. In order to get around the instability problems of $\mathcal{R}^{-1}$ there are now a number of papers on constrained optimisation and the iterative techniques. Such modifications go a long way towards nullifying the criticism of §·523, however the other points raised in §·6 still appear to be true.

Chapter 2 : Reconstruction Theory

·0 Introduction

It has already been seen in Chapter 1 that there are a number of shortcomings in many of the algorithms used for reconstructing section scans, in particular that few of them make proper allowance for the discontinuity of $\mathcal{R}^{-1}$. There are also other shortcomings in the mathematical model used. Most often it is simply assumed that

$$I(r,\phi) = I_0 \exp\left\{-\int_{L(r,\phi)} f\right\}.$$

This contains a number of tacit assumptions: e.g. that the data collected is independent of both sampling interval and collimator size - facts which are obviously incorrect.

In this chapter, consideration is given to the question "how should the scan data be processed?" This presupposes that the relation between the original section and the measured data has already been determined, and since this is not the case the chapter begins (§·1) by first developing a mathematical model to represent the scanning process. As a result it is found that the relationship between the required attenuation values and the observed data values falls naturally into four parts and a method for inventing each of these parts is developed in turn (§·2 - §·5) to give a viable algorithm for obtaining linear attenuation coefficients from section scans.

During the course of this work several constants arise (the scanner parameters of chapter 3) which represent decisions made either during the design or operation of the scanner. Consideration of these constants in the light of ideas developed in this chapter leads naturally (in chapter 3) to the question "what is the best scan data to collect?"

(Before proceeding further the reader should familiarise himself with the contents of §1·1·1 to §1·1·3 inclusive.)

·1 Mathematical model for scanner

·11 Derivation of model

A transverse body section is assigned Cartesian coordinates (x,y,z), cylindrical polar coordinates (r,ϕ,z) and origin O, Oz being perpendicular to the section (F·1). The section is scanned by a γ-ray (see chapter 1) source and detector system moving in the x-y plane. Both source and detector are collimated with a single long hole of radius a.

Assumption 1 : The beam of γ-rays is assumed to be a parallel sided cylinder of radius a.

Any line L, parallel to the plane $z = 0$, is specified by the cylindrical polar coordinates of the vector forming the perpendicular from O to L and is written $L(r,\phi,z)$ (see F·2). The position of the cylindrical beam at time t is specified by the position of its axis $L_c(r(t),\phi,0)$ (see F·3 and F·4) ($z = 0$ since scanning takes place in the x-y plane). A second set of Cartesian coordinates (ℓ,α,z_1) origin O_1, is defined (fixed with respect to the beam and moving with respect to O) such that O_1 is at $(r(t),\phi,0)$, O_1z_1 is parallel to Oz and $O_1\alpha$ is in the direction of Or (see F·4). (N.B. Since $z_1 = z$ at all times, no distinction is made between them from now on). It follows that a line parallel to, and fixed relative to, the beam is written as $L(r(t) + \alpha,\phi,z)$ relative to O (see F·4).

Consider the shaded element of F 4. The mean number of photons passing along the beam and through this element in a small time δt is

$$n\delta t \quad \frac{\delta\alpha\delta z}{\pi a^2}$$

where n is the mean number of photons/sec passing along the beam

a is the radius of the beam (collimator)

$\delta\alpha\times\delta z$ is the size of the element located at $(0,\alpha,z)$ relative to O_1.

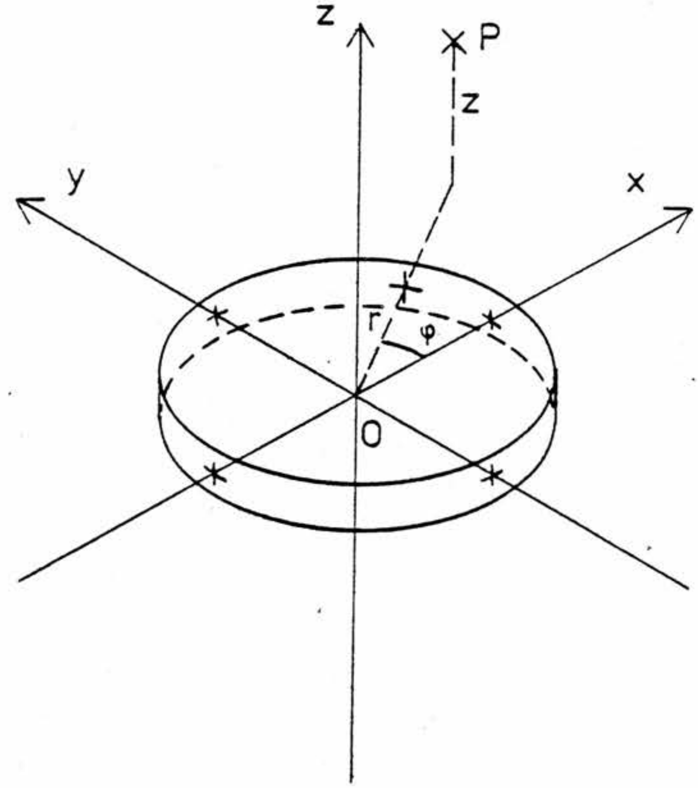


fig. 2·1 : Relation between coordinate system and section.

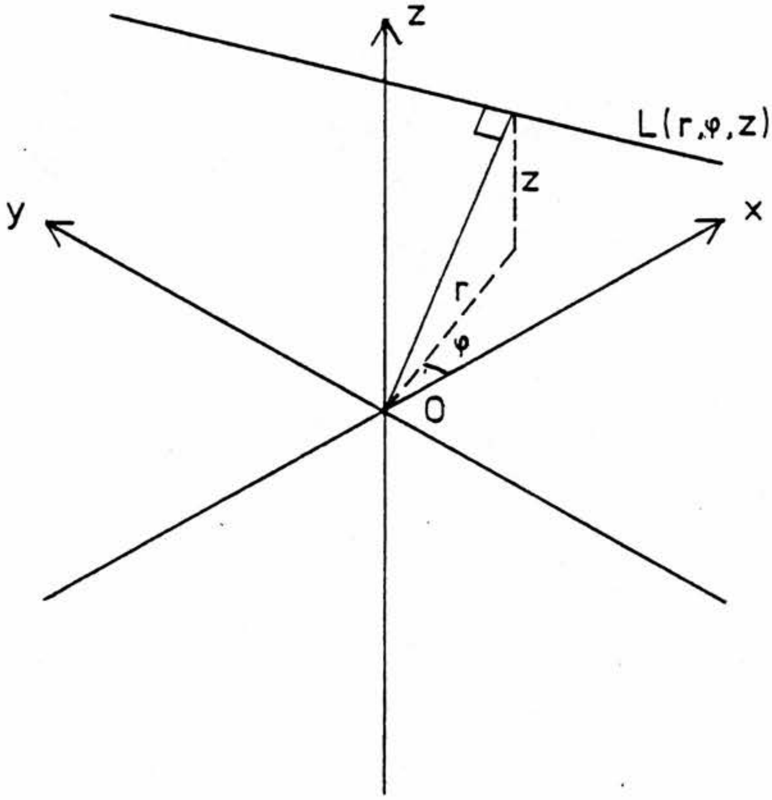


fig. 2·2 : Coordinates of line.

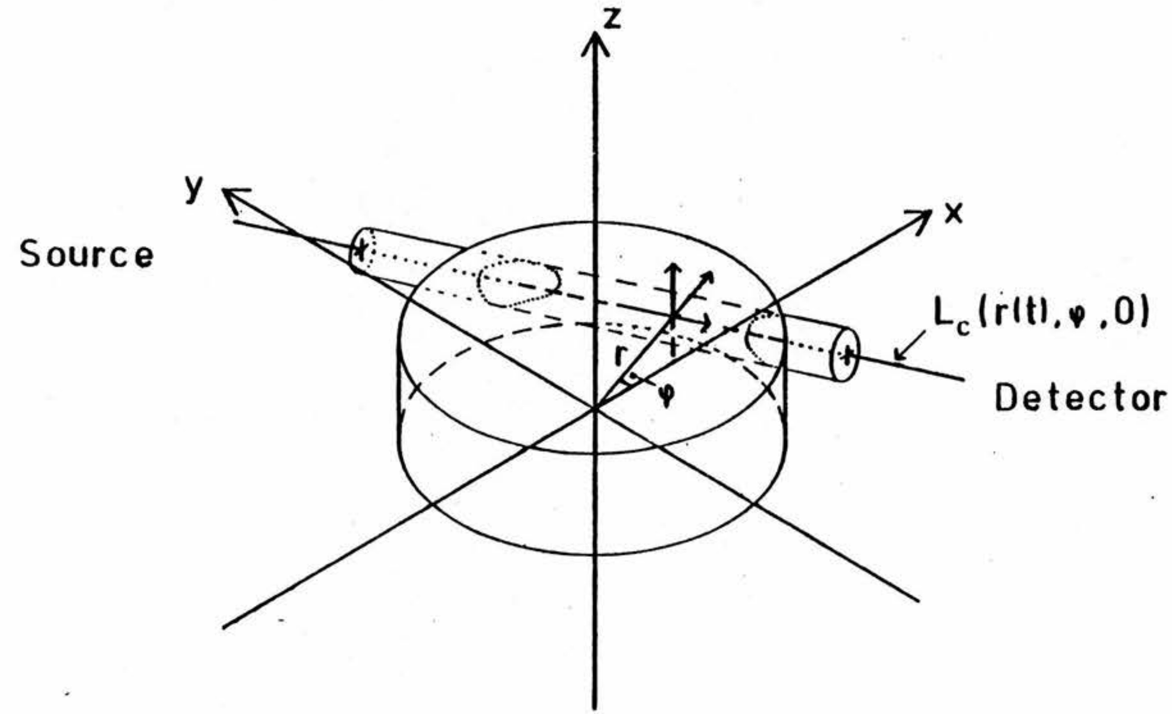


fig. 2·3 : Coordinates of beam centre.

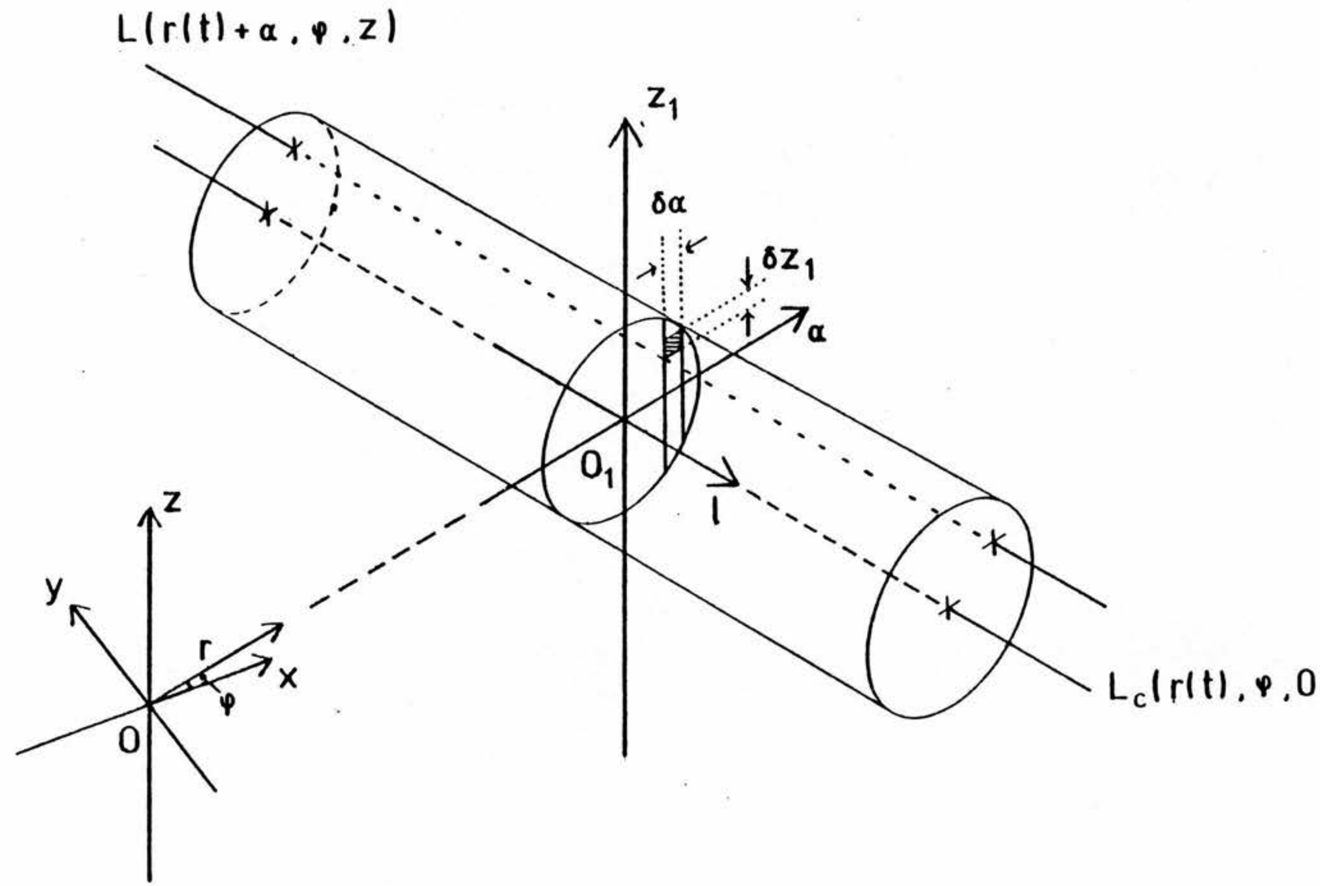


fig. 2·4 : Coordinates of beam element.

While passing from source to detector this element of the beam is attenuated by a factor

$$\exp\left\{-\int_{L(r(t)+\alpha,\,\phi,\,z)} f\right\}$$

where $f(r,\phi,z)$ is defined to be equal to the linear attenuation coefficient of the tissue at each point within the section and zero elsewhere. N.B. this last statement is true only if:-

Assumption 2: the beam is mono-energetic.

Hence the mean number of photons passing along the vertical slice of beam illustrated in F·4 and arriving at the detector is

$$\int_{-\sqrt{a^2-\alpha^2}}^{\sqrt{a^2-\alpha^2}} \frac{n\ \delta t \delta\alpha}{\pi a^2} \exp\left\{-\int_{L(r(t)+\alpha,\phi,z)} f\right\} dz \quad . \qquad \cdot 1$$

Assumption 3: the collimator size (beam size) is small relative to

$f(r,\phi,z)$ in the z direction.

With this assumption $\int_L f$ is independent of z so that E·1 can be evaluated, giving

$$\frac{n\ \delta t\ \delta\alpha}{\pi a^2}\, 2\sqrt{a^2-\alpha^2}\ \exp\left\{-\int_{L(r(t)+\alpha,\phi)}\right\}$$

Integrating this with respect to α gives the mean number of photons incident on the detector as

$$n\delta t \int_{-a}^{a} \frac{2\sqrt{a^2-\alpha^2}}{\pi a^2} \exp\left\{-\int_{L(r(t)+\alpha,\phi)}\right\} d$$

provided that

Assumption 4: scattered photons may be ignored.

or on substituting $-\alpha$ for α

$$n\,\delta t \int_{-a}^{a} \frac{2\sqrt{a^2-\alpha^2}}{\pi a^2} \exp\left\{-\int_{L(r(t)-\alpha,\phi)} f \right\} d\alpha \qquad \cdot 2$$

Defining

$$h_1(r) = \begin{cases} \dfrac{2\sqrt{a^2-r^2}}{a^2} & |r| < a \\ 0 & |r| > a \end{cases}$$

and

$$q(r,\phi) = \exp\left\{-\int_{L(r,\phi)} f \right\}.$$

E·2 may be written

$$n\,\delta t\,(h_1 *_1 q)(r(t),\phi)\ .$$

If the traverse is performed with constant velocity v then $\dot{r}(t) = v$ and $\delta t = \delta r/v$. If the domain of r is partitioned into N equal intervals of length Δr (hence forward called cells) whose mid-points are given by

$$r_j = (j - \frac{N-1}{2})\,\Delta r \quad \text{for } j = 0(1)N-1$$

then the mean number of counts μ_j observed in the j-th cell is

$$\mu_j = \int_{r_j-\frac{\Delta r}{2}}^{r_j+\frac{\Delta r}{2}} \frac{En}{v}\,(h_1 *_1 q)(r,\phi)\,dr$$

where E is the detector efficiency. Writing

$$g(r,\phi) = \frac{1}{\Delta r}\,\Pi(\frac{r}{\Delta r}) *_1 (h_1 *_1 q)(r,\phi) \qquad \cdot 3$$

and

$$\mu(r,\phi) = \ell\, g(r,\phi) \quad \text{where } \ell = \frac{En\Delta r}{v} \qquad \cdot 4$$

then

$$\mu_j = \mu(r_j,\phi). \qquad \cdot 5$$

·12 Statistical considerations

Let m_j be the observed number of counts in the j-th cell. As shown in §·11

$$E\{m_j\} = \mu_j$$

and since m_j arises from a radioactive decay process, it follows a Poisson distribution so that

$$\mathrm{var}(m_j) = \mu_j$$

and since the observed number of counts in each cell is independent

$$\mathrm{cov}(m_j, m_k) = \mu_j \delta_{jk}$$

Denote by $\underline{\underline{V}}(\underline{m})$ the $N \times N$ covariance matrix associated with the N-vector $\underline{m}$, then writing $\underline{\underline{M}}$ for diag $(\underline{\mu})$ these results may be stated

$$E\{\underline{m}\} = \underline{\mu} \qquad \underline{\underline{V}}(\underline{m}) = \underline{\underline{M}}$$

For large values of μ_j (say $\mu_j > 100$) the Poisson distribution is well approximated by the Normal distribution (Fraser, 1958 : Ch 6 §6) so for reasonably large numbers of counts/cell $\underline{m}$ is distributed $N(\underline{\mu},\underline{\underline{M}})$. ·6

·13 Summary of model

The data $\underline{m}$ observed during one traverse of a scan is a sample from an $N(\underline{\mu},\underline{\underline{M}})$ population where $\underline{\underline{M}}$ = diag $\underline{\mu}$ and $\underline{\mu}$ is related to the linear attenuation coefficient of the section by the following equations:-

$$(\underline{\mu})_j = \mu(r_j, \phi) \qquad \mu = \ell g \qquad \ell = \text{a constant}$$

$$g = h *_1 q \qquad \text{where} \quad h(r) = \frac{1}{\Delta r}\,\Pi\left(\frac{r}{\Delta r}\right) * h_1(r)$$

$$q = e^{-p}$$

$$p = \int_{L(r,\phi)} f$$

The remainder of this chapter is devoted to the following problem : Given a set of values for $\underline{m}(\phi_j)$ where $\phi_j = (j - 1)\pi/M$ and $j = 1(1)M$, estimate $f(r,\phi)$. The work proceeds by examining in turn how each of the four mappings given above may be invented.

(Note that only the mapping $f \to p$ involves r and ϕ explicitly, the remaining three mappings only involve r, ϕ appearing only in the role of a constant parameter. For this reason when discussing them explicit mention of ϕ will often be omitted, thus $g(r,\phi)$ may be written as $g(r)$.)

·2 Inversion of $\mu = \ell g$

In §·1 it was seen that

$$\mu = \ell g \quad \text{where } \ell = En\Delta r/v$$

and that the observed data $\underline{m}$ was a sample from an $N(\underline{\mu},\underline{\underline{M}})$ distribution, where $\underline{\underline{M}} = \text{diag}(\underline{\mu})$.

·21 Interpretation and estimation of ℓ

Suppose that no attenuating medium is placed between source and detector, i.e. $f(r,\phi) = 0$. Then $p = 0$, $q = e^{-p} = 1$ and

$$g = h * q = \int_{-\infty}^{\infty} h(r)dr = 1$$

Hence $(\underline{\mu})_j = \ell \;\forall j$ and m_j is a sample from an $N(\ell,\ell)$ distribution. Thus ℓ is just the expected number of counts/cell when there is no attenuating medium between source and detector. By performing scans in this manner many observations m_j of ℓ may be made and ℓ can be estimated by

$$\hat{\ell} = \frac{1}{N_1} \sum_{j=1}^{N_1} m_j \qquad \cdot 1$$

where N_1 is the number of observations. The estimate $\hat{\ell}$ is $N(\ell,\ell/N_1)$ so that ℓ may be estimated to any desired accuracy by taking N_1 sufficiently large.

·22 Estimation of g

·221 Method of estimation

The discrete counterpart of g is defined by $(\underline{g})_j = g(r_j)$ so that $\underline{g} = \ell^{-1}\,\underline{\mu}$. The estimate $\hat{\underline{g}}$ of $\underline{g}$ is therefore defined to be

$$\hat{\underline{g}} = \hat{\ell}^{-1}\,\underline{m}\ . \qquad \cdot 2$$

·222 Distribution of $\hat{\underline{g}}$ and choice of N_1

Since $\underline{m}$ is $N(\underline{\mu},\underline{\underline{M}})$, $\underline{g}$ is $N(\ell^{-1}\underline{\mu},\ell^{-2}\underline{\underline{M}})$ and

$$\hat{g} \text{ is } N(\hat{\ell}^{-1}\underline{\mu},\hat{\ell}^{-2}\underline{\underline{M}}) \qquad \cdot 3$$

The value of N_1 is now chosen so that $E(\hat{\underline{g}}) \simeq E(\underline{g})$ and $\mathrm{var}(\hat{\underline{g}}) \simeq \mathrm{var}(\underline{g})$, i.e. so that errors in $\hat{\ell}$ have negligible effect on the estimate $\hat{\underline{g}}$. Define this error by

$$\hat{\ell} = \ell + \varepsilon_\ell$$

then

$$\hat{\ell}^{-1}\underline{\mu} = \left(1 + \frac{\varepsilon_\ell}{\ell}\right)^{-1} \ell^{-1}\underline{\mu}$$

$$\simeq \left(1 - \frac{\varepsilon_\ell}{\ell}\right) \ell^{-1}\underline{\mu}$$

and

$$\hat{\ell}^{-2}\underline{\underline{M}} = \left(1 + \frac{\varepsilon_\ell}{\ell}\right)^{-2} \ell^{-2}\underline{\underline{M}}$$

$$\simeq \left(1 - \frac{2\varepsilon_\ell}{\ell}\right) \ell^{-2}\underline{\underline{M}}$$

and $\hat{\underline{g}}$ is seen to be $N\left(\left(1 - \frac{\varepsilon_\ell}{\ell}\right) \ell^{-1}\underline{\mu}\ ,\ \left(1 - \frac{2\varepsilon_\ell}{\ell}\right) \ell^{-2}\underline{\underline{M}}\right)$.

If the error in $\hat{\ell}$ had been zero (i.e. $\varepsilon_\ell = 0$) then $\hat{\underline{g}}$ would be $N(\ell^{-1}\underline{\mu},\ell^{-2}\underline{\underline{M}})$ so the effect of ε_ℓ is to introduce a bias into the estimate $\hat{\underline{g}}$. If the error introduced by ε_ℓ is to be less than some small value θ then from the mean , $\left|\frac{\varepsilon_\ell}{\ell}\right| < \theta$ for all probable values of ε_ℓ ·4

and from the variance, $\left|\frac{2\varepsilon_\ell}{\ell}\right| < \theta$ for all probable values of ε_ℓ ·5

Now $\hat{\ell}$ is $N(\ell,\ell/N_1)$ so ε_ℓ is $N(0,\ell/N_1)$ and

$$\frac{\varepsilon_\ell}{\ell} \text{ is } N(0,1/\ell N_1) \text{ and } \frac{2\varepsilon_\ell}{\ell} \text{ is } N(0,4/\ell N_1)$$

Taking the 'probable values of ε_ℓ' to be those in the $\pm 3\sigma$ range E·4 and E·5 become

$$3\sqrt{\frac{1}{\ell N_1}} < \theta \qquad \text{and} \quad 3\sqrt{\frac{4}{\ell N_1}} < \theta$$

Whence

$$N_1 > 36/\ell\theta^2 \tag*{·6}$$

It is now assumed that N_1 is chosen to fulfil this criterion and that $\hat{\ell}$ is thus a good estimate of ℓ. No further distinction is now made between $\hat{\ell}$ and ℓ.

·23 Estimation of $\underline{\underline{V}}(\hat{\underline{g}})$

From E·3 $\underline{\underline{V}}(\hat{\underline{g}}) = \hat{\ell}^{-2}\underline{\underline{M}}$ and $\underline{V}$ is therefore estimated as

$$\hat{\underline{\underline{V}}}(\hat{\underline{g}}) = \hat{\ell}^{-2}\text{diag}(\underline{m}) \quad . \tag*{·7}$$

·24 Distribution and estimation of $(\hat{\underline{g}} - \underline{g})'.(\hat{\underline{g}} - \underline{g})$

In §·3 $\hat{\underline{g}}$ will be considered in the form $\hat{\underline{g}} = \underline{g} + \underline{\varepsilon}$ and the distribution of $\underline{\varepsilon}'.\underline{\varepsilon}$ required. This is now developed. From E·3 $\hat{\underline{g}}$ is $N(\ell^{-1}\underline{\mu}, \ell^{-2}\underline{\underline{M}})$ i.e. each g_j is independent (since $\underline{\underline{M}}$ is diagonal) and is distributed $N(g_j,\ell^{-1}g_j)$. Hence

$$\varepsilon_j \text{ is } N(0,\ell^{-1}g_j)$$

and from this

$$\varepsilon_j^2 \text{ is } \ell^{-1} g_j\, \chi^2_{(1)}$$

hence

$$E\{\varepsilon_j^2\} = \ell^{-1} g_j \quad \text{and} \quad \mathrm{var}(\varepsilon_j^2) = 2(\ell^{-1} g_j)^2$$

This gives

$$E\{\ \underline{\varepsilon}'.\underline{\varepsilon}\ \} = \ell^{-1} \sum g_j$$

and

$$\mathrm{var}(\ \underline{\varepsilon}'.\underline{\varepsilon}\ ) = 2\ell^{-2} \sum g_j^2.$$

The quantities $E\{\ \underline{\varepsilon}'.\underline{\varepsilon}\ \}$ and $\mathrm{var}(\ \underline{\varepsilon}'.\underline{\varepsilon}\ )$ are thus estimated in the natural way as

$$\hat{\ell}^{-1} \sum \hat{g}_j \text{ and } 2\hat{\ell}^{-2} \sum \hat{g}_j^2$$

·3 Inversion of $g = h * q$

·31 Introduction

It was shown in §2·13 that g and q are related by $g = h * q$ ·1

where

$$h(r) = \frac{1}{\Delta r}\,\Pi\!\left(\frac{r}{\Delta r}\right) * h_1(r) \qquad \cdot 2$$

$$h_1(r) = \begin{cases} \dfrac{2\sqrt{a^2 - r^2}}{\pi a^2} & |r| < a \\ 0 & |r| > a \end{cases}$$

and in §2·2 that g is known only from its discrete estimate $\hat{\underline{g}}$ which is contaminated by statistical errors, that is

$$\hat{\underline{g}} = \underline{g} + \underline{\varepsilon} \qquad \cdot 3$$

where

$$(\underline{g})_j = g(r_j)$$

and

$$\underline{\varepsilon} \text{ is } N(0, \ell^{-1}\,\mathrm{diag}(\underline{g})) \qquad \cdot 35$$

and it is from this information that an estimate $\hat{\underline{q}}$ of q is to be made.

However, there is a fundamental difficulty inherent in inverting a convolution equation. Consider E·1 where the functions g, h and q are defined on a continuum, this may be written as a Fredholm integral equation of the first kind

$$g(r) = \int K(r,\alpha)\,q(\alpha)\,d\alpha \qquad \cdot 4$$

where the kernel

$$K(r,\alpha) = h(r - \alpha).$$

Since $h \in \mathcal{L}^1$, it follows from the Riemann-Lebesgue Theorem that, even if the inverse does exist, it is discontinuous (see (Miller, 1974:p 176), (Phillips

1962:p 84) and (Taylor, 1958:p 86 Jl 3·1-B)), so the presence of errors in the measurements of g presents a serious problem. As is to be expected, remodelling the equation in Fourier space (Hunt, 1970:p 166) as

$$G = HQ$$

or in discrete form as a matrix/vector product (Hunt, 1972a)

$$\underline{g} = \underline{\underline{h}}\,\underline{q}$$

only changes the appearance of the difficulty but does not remove it.

Miller (Miller, 1974) has discussed the inversion of E·4 for arbitary kernels and a number of authors (Phillips, 1962),(Twomey, 1965), (Hunt, 1970, 1971,1972) and others) have considered the particular case of deconvolution, in each case using a constrained optimisation approach. Hunt (Hunt, 1972b) has also considered the solution by Wiener filtering and discussed its relation to the constrained optimisation solution. For the purposes of solving E·1, the constrained optimization technique given by Hunt (Hunt, 1971) is used, an account of it being given in §·32 and Appendix 2·2.

There is a fundamental decision to be made concerning the method used to solve E·1. It will be noted that the model (E·1) is defined on a continuum whereas the solution $\underline{q}$ can only be given at discrete points. One can therefore develop solutions in two ways:-

1) develop a solution using functions defined on a continuum (which means the use of calculus of variations) and then give a discrete version of this.
2) give a discrete counterpart to equation E·1 and then develop a discrete solution (using non-linear programming).

It is the second choice which is followed here.

·32 Deconvolution

In Appendix 2·1, a discrete version of E·1 is given and is

$$\underline{g} = \underline{\underline{h}}^{*}\underline{q} \qquad \cdot 41$$

where

$$(\underline{g})_j = g(r_j), (\underline{q})_j = q(r_j)$$

$$(\underline{\underline{h}}^{*})_{jk} = h^{*}_{(j-k)|N_2}$$

$$h^{*}_j = \begin{cases} \Delta r \quad h_j & 0 \le j \le N_2/2 \\ \Delta r \quad h_{j-N_2} & N_2/2 \le j \le N_2 - 1 \end{cases}$$

$$h_j = h(j\Delta r)$$

Combining this with E·3 gives the model

$$\underline{\hat{g}} = \underline{\underline{h}}^{*}\underline{q} + \underline{\varepsilon} \qquad \cdot 43$$

and this is used as the starting point in deriving a solution. A sequence $\underline{q}$ is called a 'feasible solution' to E·43 if

$$(\underline{\hat{g}} - \underline{\underline{h}}^{*}\underline{q})'(\underline{\hat{g}} - \underline{\underline{h}}^{*}\underline{q}) \le \underline{\varepsilon}'\underline{\varepsilon} = e^2$$

that is, if $\underline{\underline{h}}^{*}\underline{q}$ is close to $\underline{\hat{g}}$. Out of all these solutions, one is chosen which is smoothest, in the sense that $\underline{q}$ is chosen to satisfy

$$\min((\underline{\underline{c}}\ \underline{q})'(\underline{\underline{c}}\ \underline{q}))$$

where $\underline{\underline{c}}$ is the circulant matrix associated with the sequence (see §1·1·15)

$$\underline{c} = (-2,1,0,0,\ldots,0,1)'$$

The product $\underline{\underline{c}}\ \underline{q}$ is the convolution of $\underline{\hat{q}}$ with the sequence $\underline{c}$ and gives the second difference of $\underline{q}$ so that this minimisation requirement is the discrete version of

$$\min \left| \frac{\partial^2 q}{\partial r^2} \right|^2 .$$

The problem of solving E·43 is now formulated as a constrained optimisation problem

$$\min \{(\underline{\underline{c}}\ \underline{q})'(\underline{\underline{c}}\ \underline{q})\} \quad \text{subject to} \quad (\hat{\underline{g}} - \underline{\underline{h}}^* \underline{q})'(\hat{\underline{g}} - \underline{\underline{h}}^* \underline{q}) \le e^2$$

From the Convolution Theorem (§1·1·27) it follows that the D.F.T.'s of $\underline{\underline{c}}\ \underline{q}$ and $\hat{\underline{g}} - \underline{\underline{h}}^*\underline{q}$ are $\sqrt{N_2}\ \underline{\underline{C}}\ \underline{Q}$ and $\hat{\underline{G}} - \sqrt{N_2}\ \underline{\underline{H}}^*\underline{Q}$. Thus by applying Parseval's Theorem (§1·1·28) the optimisation problem becomes

$$\min \{N_2\ \underline{Q}^T \underline{\underline{C}}^T \underline{\underline{C}}\ \underline{Q}\} \quad \text{subject to} \quad (\underline{G} - \sqrt{N_2}\ \underline{\underline{H}}^*\underline{Q})^T(\hat{\underline{G}} - \sqrt{N_2}\ \underline{\underline{H}}^*\underline{Q}) \le e^2$$

Taking the Lagrangian L to be

$$L = N_2 \underline{Q}^T \underline{\underline{C}}^T \underline{\underline{C}}\ \underline{Q} + \lambda[e^2 - (\hat{\underline{G}} - \sqrt{N_2}\ \underline{\underline{H}}^*\underline{Q})^T(\hat{\underline{G}} - \sqrt{N_2}\underline{\underline{H}}^*\underline{Q})]$$

then applying §2·2·3 Th·8 gives $\nabla L = 0$ which leads to

$$N_2 \underline{\underline{C}}^T \underline{\underline{C}}\ \underline{Q} + \lambda(-\sqrt{N_2}\underline{\underline{H}}^*)^T(\hat{\underline{G}} - \sqrt{N_2}\ \underline{\underline{H}}^*\underline{Q}) = 0 \quad \text{where} \quad \lambda \ge 0$$

and hence

$$\sqrt{N_2}(\lambda \underline{\underline{H}}^{*T}\underline{\underline{H}}^* + \underline{\underline{C}}^T\underline{\underline{C}})\underline{Q} = \lambda \underline{\underline{H}}^{*T}\hat{\underline{G}}$$

The case $\lambda = 0$ corresponds to the constraint being inactive and is assumed not to arise. Then $\lambda > 0$. Putting $\gamma = \lambda^{-1}$ and using the fact that $\underline{\underline{H}}^{*T}\underline{\underline{H}}^* + \gamma\underline{\underline{C}}^T\underline{\underline{C}}$ is non-singular (see App 2·3)

$$\underline{Q} = \frac{1}{\sqrt{N_2}}(\underline{\underline{H}}^{*T}\underline{\underline{H}}^* + \gamma\underline{\underline{C}}^T\underline{\underline{C}})^{-1}\underline{\underline{H}}^{*T}\hat{\underline{G}} \qquad \cdot 5$$

Note that the matrices $\underline{\underline{H}}^*$ and $\underline{\underline{C}}$ are diagonal so that E·5 may be written in terms of components as

$$Q_k = \frac{1}{\sqrt{N_2}} \quad \frac{\bar{H}_k^* \hat{G}_k}{\bar{H}_k^* H_k^* + \gamma \bar{C}_k C_k} \quad . \qquad \cdot 52$$

The estimate $\hat{\underline{q}}$ of q is now defined by

$$\hat{\underline{q}} = \underline{\underline{W}}^{-1}\hat{\underline{Q}} \qquad \cdot 53$$

and $\hat{\underline{Q}}$ is defined by E·5 (or E·52).

By substituting E·5 in condition 2 of §2·2·3 Th·8 and using the facts that $\lambda > 0$ and diagonal matrices commute one obtains

$$\underline{\hat{G}}^T \left[(\underline{\underline{H}}^{*T}\underline{\underline{H}}^* + \gamma\underline{\underline{C}}^T\underline{\underline{C}})^{-1}(\gamma\underline{\underline{C}}^T\underline{\underline{C}})\right]^2 \underline{\hat{G}} = e^2 \qquad \cdot 55$$

or in terms of components

$$\sum_{k=0}^{N_2-1} \left(\frac{\gamma\overline{C}_k C_k}{\overline{H}_k^* H_k^* + \gamma\overline{C}_k C_k} \right)^2 \overline{\hat{G}}_k \hat{G}_k = e^2 \qquad \cdot 6$$

from which γ may be obtained. Having obtained γ (see §·35) it may be substituted in E·5 (or E·52) to obtain $\hat{\underline{Q}}$ and thence using E·53 to obtain $\hat{\underline{q}}$.

·321 Choice of N_2

In §·2 the number of observed data points was defined to be N, where N is any positive integer. The design of the scanner described in §4 and of the 'front end' software (see §5) to be used for data collection implies that N is even and that there are the same number of cells on either side of the centre of rotation. Since the deconvolution is to be performed using an F.F.T. algorithm it is necessary that the number of data points is an integer power of 2. The value of N_2 is therefore defined by

$$N_2 = \min \{2^n : 2^n \geq N \text{ and } n \in \mathbb{N}\}$$

and the number of data values is artificially increased from N to N_2 by the software by padding out with $(N_2 - N)/2$ cells before and after the original N cells. Each of these is assigned a value $\hat{\ell}$ (since it is assumed that they represent measurements taken beyond the edge of the section) and a variance and covariance of 0.

The vectors $\underline{m}, \underline{\mu}$ and matrix $\underline{\underline{M}}$ are now used to denote either the original N and $N \times N$ quantities or the supplemented N_2 and $N_2 \times N_2$

quantities and suitable comment is made to indicate which is under discussion on any given occasion. (In practice it is usually obvious simply by noting whether the associated indices are running from 0 to N - 1 or 0 to N_2 - 1).

·33 Existence and usefullness of real space solutions

The reason for calculating the solutions in Fourier space is easily seen. Applying §1·1·22 to E·5 and E·55 gives

$$\hat{\underline{g}} = (\underline{\underline{h}}^{*'}\underline{\underline{h}}^{*} + \gamma\underline{\underline{c}}'\underline{\underline{c}})^{-1}\underline{\underline{h}}^{*'}\hat{\underline{g}} \qquad \cdot 7$$

and

$$\hat{\underline{g}}' \, [(\underline{\underline{h}}^{*'}\underline{\underline{h}}^{*} + \gamma\underline{\underline{c}}'\underline{\underline{c}})^{-1} \, \underline{\underline{c}}'\underline{\underline{c}}]^2 \, \hat{\underline{g}} = e^2 \qquad \cdot 8$$

from which solutions could be calculated as before. However the matrices of E·5 and E·55 are diagonal while those of E·7 and E·8 are not, thus calculation of $(\underline{\underline{H}}^{*T}\underline{\underline{H}}^{*} + \gamma\underline{\underline{C}}^{T}\underline{\underline{C}})^{-1}$ only involves taking reciprocals of diagonal elements whereas calculating $(\underline{\underline{h}}^{*'}\underline{\underline{h}}^{*} + \gamma\underline{\underline{c}}'\underline{\underline{c}})^{-1}$ involves matrix inversion which is considerably more time consuming.

·34 Uniqueness of solution

Note that, with the assumptions given, E·5 and E·6 also imply the uniqueness of the solution. Assuming that the constraint is active ($\lambda \neq 0$) and using §2·2·3 Th·8 ($\lambda \geq 0$) gives $\gamma > 0$. But for $\gamma > 0$, the left hand side of E·6 is a strictly increasing function of γ . Thus if there exist two solutions γ_1 , $\hat{\underline{Q}}_1$ and γ_2, $\hat{\underline{Q}}_2$ then E·6 implies $\gamma_1 = \gamma_2$ and substituting these in E·5 implies $\hat{\underline{Q}}_1 = \hat{\underline{Q}}_2$.

·35 Calculation of e^2

In §·32 a value is required for $e^2 = \underline{\varepsilon}'\underline{\varepsilon}$ but all that is known of $\underline{\varepsilon}$ and hence e^2 is its distribution (see §·24). It is shown that

$$E\{\underline{\varepsilon}'\underline{\varepsilon}\} = \ell^{-1} \sum_j g_j \qquad \mathrm{var}(\underline{\varepsilon}'\underline{\varepsilon}) = 2\ell^{-2} \sum_j g_j^2$$

The quantity e^2 is therefore estimated by

$$\hat{e}^2 = \ell^{-1} \sum \hat{g}_j$$

and E·6 of §·32 used iteratively by looking for a value of γ such that

$$\left| \hat{e}^2 - \sum \left(\frac{\gamma \bar{C}_k C_k}{\bar{H}_k^* H_k^* + \gamma \bar{C}_k C_k} \right)^2 \bar{\hat{G}}_k \hat{G}_k \right| \le 2\sqrt{2}\,\ell^{-1} \sqrt{\sum \hat{g}_j^2}$$

i.e. when the left hand side of E·6 is within two standard deviations of $\hat{e}^2$.

·36 Evaluation of $\underline{\underline{H}}^*$

From E·2 h(r) is defined as the convouution of two functions. Thus evaluating $\underline{\underline{H}}^*$ by first evaluating $\underline{h}$ and hence $\underline{h}^*$ involves numerical integration to obtain $\underline{h}$. However H can be evaluated analytically from h and then sampled directly to give $\underline{H}^*$. This is the method adopted here.

It is straightforward to show that the C.F.T. of

$$\frac{1}{\Delta r} \Pi \left(\frac{r}{\Delta r} \right) \qquad \text{is} \qquad \frac{\sin(\pi \Delta r R)}{\pi \Delta r R}$$

and that the C.F.T. of $h_1(r)$ is $2\,\dfrac{J_1(2\pi a R)}{2\pi a R}$

see (Erdélyi et al, 1954; I §1·3·8). Combining these results gives

$$H(R) = 2\,\frac{\sin(\pi \Delta r R)}{\pi \Delta r R}\;\frac{J_1(2\pi a R)}{2\pi a R} \qquad \cdot 9$$

From §2·1·5

$$H_k^* = \frac{1}{\sqrt{N_2}} \{H(k\Delta R) + H((k - N_2)\Delta R)\}$$

provided $H(R) \simeq 0 \;\forall\; |R| \ge N_2 \Delta R/2 = (2\Delta r)^{-1}$ so H_k^* is evaluated as

$$H_k^* = \begin{cases} \dfrac{1}{\sqrt{N_2}} H(k\Delta R) & 0 \le k \le N_2/2 \\ H^*_{N_2-k} & N_2/2 < k \le N_2 - 1 \end{cases}$$

Conditions on a and Δr to ensure that $H(R) \simeq 0 \quad \forall |R| \ge N_2 \; \Delta R/2$ are examined in §3·2.

·37 Statistical considerations

·371 Expected value of $\hat{\underline{q}}$

From §·222 $\hat{\underline{g}}$ is $N(\ell^{-1}\underline{\mu}, \ell^{-2}\underline{\underline{M}})$ i.e. $N(\underline{g}, \ell^{-1}\text{diag}(\underline{g}))$ hence

$E\{\hat{\underline{g}}\} = \underline{g} = \underline{\underline{h}}^* \underline{q}$

Define $\underline{\underline{\ell}}$ by $\underline{\underline{\ell}} = (\underline{\underline{h}}^{*'}\underline{\underline{h}}^* + \gamma\underline{\underline{c}}'\underline{\underline{c}})^{-1}\underline{\underline{h}}^{*'}$

$$\begin{aligned} \text{then} \quad E\{\hat{\underline{q}}\} &= E\{\underline{\underline{\ell}}\hat{\underline{g}}\} \qquad\qquad \text{from E·7} \\ &= \underline{\underline{\ell}}\, E\{\underline{g}\} \\ &= \underline{\underline{\ell h}}^* \underline{q} \\ &\ne \underline{q} \end{aligned}$$

since $\underline{\underline{\ell h}}^*$ is the identity matrix only if $\gamma = 0$ and this is not the case (see §·32). Thus $\hat{\underline{q}}$ is a biased estimate of $\underline{q}$ (see Hunt, 1971: p 288) for further discussion.

·372 Estimation and calculation of $\underline{\underline{V}}(\hat{\underline{q}})$

Since $\hat{\underline{q}} = \underline{\underline{\ell}}\hat{\underline{g}}$ it follows immediately that

$\underline{\underline{V}}(\hat{\underline{q}}) = \underline{\underline{\ell}}\;\underline{\underline{V}}\;(\hat{\underline{g}})\underline{\underline{\ell}}'$

the covariance matrix of $\hat{\underline{q}}$ is therefore estimated as

$\hat{\underline{\underline{V}}}(\hat{\underline{q}}) = \underline{\underline{\ell}}\;\hat{\underline{\underline{V}}}(\hat{\underline{g}})\underline{\underline{\ell}}'$

$$= \ell^{-2}\underline{\underline{\ell}}\ \mathrm{diag}(\underline{m})\underline{\underline{\ell}}'$$ from §·23 E·7

This raises the question of evaluating $\underline{\underline{\ell}}$. Elementary application of the theorem in §1·1·21 to the definition of $\underline{\underline{\ell}}$ gives

$$\underline{\underline{L}} = \frac{1}{N_2}\,(\underline{\underline{H}}^{*T}\underline{\underline{H}}^{*} + \gamma\underline{\underline{C}}^{T}\underline{\underline{C}})^{-1}\underline{\underline{H}}^{*T}$$

Since $\underline{\underline{H}}^{*}, \underline{\underline{C}}$ are diagonal so is $\underline{\underline{L}}$, hence $\underline{\underline{\ell}}$ is circulant and if the first column of $\underline{\underline{\ell}}$ is known, all its elements are known. But the first column of $\underline{\underline{\ell}}$ is $\underline{\ell} = \underline{\underline{W}}^{-1}\underline{L}$ (where $\underline{L}$ is just the diagonal elements of $\underline{\underline{L}}$) and this may be calculated by an F.F.T. The matrix $\underline{\underline{\ell}}$ may therefore be found without any need for matrix multiplication. (Using the facts that $\underline{H}^{*} = \underline{\underline{W}}\underline{h}^{*}$ and $\underline{C} = \underline{\underline{W}}\,\underline{c}$ where $\underline{h}^{*}$ and $\underline{c}$ are real, together with §1·1·23 shows that $\underline{\ell}$ and hence $\underline{\underline{\ell}}$ is real).

N.B. There is in §·37 the tacit assumption that γ is a constant which is statistically independent of the data, a fact which is clearly dubious. It does not appear to be possible to develop the results of this section in a way which makes allowance for possible statistical variations in γ, so that the validity of these results must rest on a demonstration that for all practical purposes the statistical variations of γ may be neglected.

·4 Inversion of $q = e^{-p}$

It was shown in §·13 that q and p are related by

$$q = e^{-p} \qquad \cdot 1$$

and in §·37 that q is known only from its discrete biased estimate $\hat{\underline{q}}$.

·41 Estimation of p

It follows from E·1 that

$$(\underline{p})_j = p(r_j) = -\ln(q(r_j)) = -\ln((\underline{q})_j)$$

which is conveniently abbreviated to $\underline{p} = -\ln \underline{q}$. The function p is therefore estimated by

$$\hat{\underline{p}} = -\ln \hat{\underline{q}} \; .$$

Note that since $E\{\hat{\underline{q}}\} \neq \underline{q}$ (see §·371) then $E\{\hat{\underline{p}}\} \neq \underline{p}$.

·42 Estimation of $\underline{\underline{V}}(\hat{\underline{p}})$

Suppose x is such that $E\{x\} = \mu_x$, $\mathrm{var}(x) = \sigma_x^2$ and $\sigma_x \ll \mu_x$. For any x define a value y by $y = -\ln x$ so that $E(y) \simeq -\ln \mu_x$. Since $\sigma_x \ll \mu_x$ i.e. $|x - \mu_x|$ is probably small then $y = -\ln x$ will be well approximated by the linear approximation

$$y - (-\ln \mu_x) = -\frac{1}{\mu_x}(x - \mu_x)$$

Applying this to each component of $\underline{p}$ and $\underline{q}$ the relation $\underline{p} = -\ln \underline{q}$ can be approximated by

$$p_j - (-\ln \mu_{q_j}) = -\frac{1}{\mu_{q_j}}(q_j - \mu_{q_j})$$

but, bearing in mind that for the linear transformation $\underline{x} \rightarrow \underline{\underline{A}}\,\underline{x} + \underline{b}$ $\underline{\underline{V}}(\underline{\underline{A}}\,\underline{x} + \underline{b}) = \underline{\underline{A}}\,\underline{\underline{V}}(\underline{x})\underline{\underline{A}}'$, then $\underline{\underline{V}}(\hat{p})$ can be approximated by

$$\underline{\underline{V}}(\underline{p}) \simeq \mathrm{diag}(-\underline{\mu}_q)^{-1}\,\underline{\underline{V}}(\underline{q})\,\mathrm{diag}(-\underline{\mu}_q)^{-1} \qquad \cdot 2$$

The matrix $\underline{\underline{V}}(\hat{\underline{p}})$ will therefore be estimated by

$$\hat{\underline{\underline{V}}}(\hat{\underline{p}}) = \mathrm{diag}(\hat{\underline{q}})^{-1}\hat{\underline{\underline{V}}}(\hat{\underline{q}})\;\mathrm{diag}(\hat{\underline{q}})^{-1} \qquad \cdot 3$$

from §·37

$$\hat{\underline{q}} = \underline{\underline{\ell}}\;\hat{\underline{g}} = \ell^{-1}\underline{\underline{\ell}}\underline{m}$$

and

$$\hat{\underline{\underline{V}}}(\hat{\underline{q}}) = \ell^{-2}\;\underline{\underline{\ell}}\;\mathrm{diag}(\underline{m})\underline{\underline{\ell}}' \; .$$

Substituting these in E·3 gives

$$\hat{\underline{\underline{V}}}(\hat{\underline{p}}) = \mathrm{diag}(\underline{\underline{\ell}}\ \underline{m})^{-1}\ \underline{\underline{\ell}}\ \mathrm{diag}(\underline{m})\,\underline{\underline{\ell}}'\ \mathrm{diag}(\underline{\underline{\ell}}\ \underline{m})^{-1} \qquad \cdot 5$$

When making any use of E·5 it should not be forgotten just how questionable the result is, as the transition from E·2 to E·3 involves the assumption that the estimate of a sample from a distribution is a good substitute for the mean of the distribution, and when the estimate is known to be biased anyway.

·5 Inversion of $p = \mathcal{R}f$

·51 Introduction

It has already been seen in §1·1·48 that $\mathcal{R}^{-1}$ is discontinuous so that any attempt to find f by using the equation

$$f = \mathcal{R}^{-1}p$$

is likely to produce highly unpredictable results unless some form of low pass filtering is incorporated in the solution in order to control the effects of noise in the data. The decision to incorporate such filtering immediately, raises the question, "What cut-off frequency should be used, and what shape of filter?" In this section the relationship between the density function f and $\hat{p}$ its estimated or observed Radon transform is taken as

$$\hat{p} = \mathcal{R}f + \varepsilon \qquad \cdot 1$$

where ε is some error function. This relationship is solved by reformulating the problem as

"find f such that $S(f)$ is minimised subject to the constraint
$d(\hat{p}, \mathcal{R}f) \leq |e|$" ·2

where $S(f)$ is a measure of the 'smoothness' of f and d is a measure of 'distance' between $\hat{p}$ and $\mathcal{R}f$. The solution to this new problem is denoted by $\hat{f}$ (N.B. $\hat{f} \neq \mathcal{R}^{-1}\hat{p}$ in general).

It is as well to note that in reformulating the problem as E·2 two major assumptions have already been made:-

1) the function $\hat{p}$ has been regarded from the traditional point of view as an essentially correct estimate of p with a few small errors which need to be allowed for; one could however regard $\hat{p}$ as a member of an ensemble in which case the reconstruction problem would be formulated using a probability metric as follows:- let I denote the operator used to calculate the estimate $\hat{f}$ of f, then I is chosen in order to

$$\text{minimise } E\{\, \|f - I\hat{p}\|_{B^2} \}$$

and one has a 'Wiener filtering' type of problem. This approach is not considered further.

2) the functions p, f, S and d have been assumed to be defined on a continuum. At first sight it appears that there is the choice, as in §·23, of whether to start with p, f, S and d defined on a continuum, use the Calculus of Variations to develop a solution defined on a continuum and then write down a discrete version of it or whether to start with a discrete version of the problem and use non-linear optimization theory to develop a discrete solution. However, when dealing with non-parametric solutions, this is not so, as use is made of the theorem $\mathcal{P}\mathcal{F}_2 = \mathcal{F}_1\mathcal{R}$ which has no discrete counterpart thus forcing one to work on a continuum. (This is not the case for parametric solutions. In this situation one can develop solutions based on non-linear optimization by using the results of Appendix 2·2).

In this section, non-parametric solutions will be developed for two different functions S and two different functions d, namely

$$S(f) = \|f\|^2_{B^2} \quad \text{and} \quad S(f) = \|\nabla^2 f\|^2_{B^2}$$

and

$$d(\hat{p},\mathcal{R}f) = ||\hat{p} - \mathcal{R}f||_{B^1 \times S^1} \quad \text{and} \quad d(\hat{p},\mathcal{R}f) = ||\mathcal{P}^{-1}\mathcal{F}_1\hat{p} - \mathcal{F}_2 f||_{\mathbb{R}^2}$$

where the density function and its projections will be considered as elements from $\mathcal{L}^2(B^2)$ and $\mathcal{R}(\mathcal{L}^2(B^2))$ respectively (see §1·1·4 for notation).

Note: (1) Frequent use is made of the projection theorem $\mathcal{P}\mathcal{F}_2 = \mathcal{F}_1\mathcal{R}$. However since the operator $\mathcal{P}$ represents what is basically a change of conceptual view point rather than the assignment of new numerical values, its explicit mention will usually be omitted.

(2) In §1·1·4 it was assumed without loss of generality that the support of f was contained in the unit ball, B^n. Since, in practice, the function f will have a support contained in $|r| \le D/2$ a symbol is needed to denote the ball with this radius. As no problems will be found to arise, this will be denoted by the same symbol.

·52 Solution of modified reconstruction problem

·521 Case 1 : $S(f) = ||f||^2_{B^2}$, $d(\hat{p},\mathcal{R}f) = ||\hat{p} - \mathcal{R}f||_{B^1 \times S^1}$

From Parseval's theorem in two dimensions

$$S(f) = ||f||^2_{B^2} = ||\mathcal{F}_2 f||^2_{\mathbb{R}^2} = \frac{1}{2}\int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dR\ |R|\,|\mathcal{F}_2 f|^2 \qquad \cdot 11$$

and from Parseval's theorem in one dimension

$$d(\hat{p},\mathcal{R}f)^2 = ||\hat{p} - \mathcal{R}f||^2_{B^1 \times S^1} = ||\mathcal{F}_1(\hat{p} - \mathcal{R}f)||^2_{\mathbb{R} \times S^1}$$

$$= \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dR |\mathcal{F}_1\hat{p} - \mathcal{F}_1\mathcal{R}f|^2$$

$$= \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dR |\mathcal{F}_1\hat{p} - \mathcal{F}_2 f|^2 \qquad \cdot 12$$

since $\mathcal{F}_2 = \mathcal{F}_1\mathcal{R}$ from the Projection Theorem (§1·1·444). Finding $\hat{f}$ is now a case of solving the problem

$$\text{"minimise} \int_0^{2\pi} d\Theta \int_\infty^\infty dR|R| \, |\mathcal{F}_2 f|^2 \text{ subject to the constraint} \qquad \cdot 13$$

$$\int_0^{2\pi} d\Theta \int_\infty^\infty dR \, |\mathcal{F}_1\hat{p} - \mathcal{F}_2 f|^2 \le e^2 \text{"} \qquad \cdot 14$$

(N.B. the multiplicative constant $\frac{1}{2}$ has been dropped from S, and the constraint condition squared, neither of these alters the solution which will be obtained.)

Denote by $\hat{f}$ the solution to E·13 and E·14 and by $\mathcal{F}_2\hat{f}$ and $\mathcal{R}\hat{f}$ the associated functions in Fourier space and Projection space respectively. Note that $\hat{p}$ is an estimate of p based on the observed data, that $\hat{f}$ is an estimate of f based on $\hat{p}$ and that $\mathcal{R}\hat{f} \neq \hat{p}$ in general (equality occurs only if $e^2 = 0$, i.e. for noise free data). This problem is solved for the function $\mathcal{F}_2\hat{f}$ and $\hat{f}$ is then deduced in terms of $\mathcal{F}_2\hat{f}$.

Solving E·13 and E·14 for $\mathcal{F}_2 f$ is the form of problem treated by the Calculus of Variations (see Gelfand, 1963: p 45 equation 35) whence one arrives at

$$|R|\mathcal{F}_2 f - \lambda_c(\mathcal{F}_1\hat{p} - \mathcal{F}_2 f) = 0$$

(where λ_c (c for continuous) is a constant yet to be determined) and hence

$$\mathcal{F}_2 f = \frac{\lambda_c}{|R| + \lambda_c} \mathcal{F}_1\hat{p} \quad . \qquad \cdot 15$$

This is taken as the estimate $\mathcal{F}_2\hat{f}$. Applying $\mathcal{F}_2^{-1}$ to both sides of E·5

$$\hat{f} = \mathcal{F}_2^{-1} \left(\frac{\lambda_c}{|R| + \lambda_c} \right) \mathcal{F}_1\hat{p} \quad . \qquad \cdot 16$$

This solution is now manipulated into a form more suitable for the later discrete development of §·53 and §3· . From the Projection Theorem $\mathcal{F}_2^{-1} = \mathcal{R}^{-1}\mathcal{F}_1^{-1}$ hence

$$\hat{f} = \mathcal{R}^{-1}\mathcal{F}_1^{-1} \left(\frac{\lambda_c}{|R| + \lambda_c} \mathcal{F}_1\hat{p} \right)$$

and from §1·1·462

$$\mathcal{R}^{-1}(p) = \mathcal{B}\mathcal{F}_1^{-1}(|R|\mathcal{F}_1 p) \qquad \cdot 17$$

hence

$$\begin{aligned}\hat{f} &= \mathcal{B}\mathcal{F}_1^{-1}\left\{|R|\mathcal{F}_1\mathcal{F}_1^{-1}\left(\frac{\lambda_c}{|R| + \lambda_c}\mathcal{F}_1\hat{p}\right)\right\} \\ &= \mathcal{B}\mathcal{F}_1^{-1}\left(\frac{\lambda_c|R|}{|R| + \lambda_c}\mathcal{F}_1 p\right)\end{aligned} \qquad \cdot 18$$

Note the change brought about by the reformulation of the problem. When evaluating $\mathcal{R}^{-1}(\hat{p})$ using E·17 (the most common starting point for inverting $\mathcal{R}$) one starts by multiplying the Fourier transform of $\hat{p}$ $(\mathcal{F}_1\hat{p})$ by $|R|$, it is therefore not surprising that the discret analogue of this is sensitive to statistical errors in $\hat{p}$. In the modified formulation of the problem, $\mathcal{F}_1\hat{p}$ is multiplied by

$$\frac{\lambda_c\ |R|}{|R| + \lambda_c}$$

which is bounded above by λ_c and the excessive boosting of high frequency components in $\hat{p}$ is restrained.

The constant λ_c can be evaluated by substituting E·15 into E·14 giving

$$\int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dR \left| \frac{|R|}{|R| + \lambda_c}\mathcal{F}_1\hat{p}\right|^2 = e^2 \qquad \cdot 19$$

Note the assumption that the constraint is active, i.e. that $\leq e^2$ can be replaced by $= e^2$. If this is not the case then $\mathcal{F}_2\hat{f}$ would be an unconstrained solution of minimise $S(f)$. By inspection of E·13 this would mean that $\mathcal{F}_2\hat{f} = 0$ and $\hat{f} = 0$, and hence that the data $\mathcal{F}_1\hat{p}$ was not significantly different from zero. It is assumed that this is not the case.

·522 Case 2 : $S(f) = \|\nabla^2 f\|^2_{B^2}$, $d(\hat{p},\mathcal{R}f) = \|\hat{p} - \mathcal{R}f\|_{B^1\times S^1}$

The development in this section precisely parallels that of §·521, the definitions of S and d are used to reformulate the problem in Fourier space and the Calculus of Variations applied.

From Parseval's theorem and the theorems for the C.F.T. of derivatives

$$S(f) = ||\nabla^2 f||^2_{B^2} = ||-4\pi^2 R^2 \mathcal{F}_2 f||^2_{\mathbb{R}^2}$$

$$= 8\pi^4 \int_0^{2\pi} d\Theta \int_{-\infty}^{\infty} dR |R|^5 |\mathcal{F}_2 f|^2 \qquad \cdot 21$$

and as in §·521

$$d(\hat{p}, \mathcal{R}f)^2 = \int_0^{2\pi} d\Theta \int_{-\infty}^{\infty} dR\ |\mathcal{F}_1 \hat{p} - \mathcal{F}_2 f|^2 \le e^2 \qquad \cdot 22$$

Applying the Calculus of Variations to solve "min S subject to $d^2 \le e^2$" gives

$$|R|^5 \mathcal{F}_2 f - \lambda_c (\mathcal{F}_1 \hat{p} - \mathcal{F}_2 f) = 0$$

(where λ_c is a constant yet to be determined) hence

$$\mathcal{F}_2 f = \frac{\lambda_c}{|R|^5 + \lambda_c} \mathcal{F}_1 \hat{p} \qquad \cdot 23$$

and following the same reasoning as §·521 leads to

$$\hat{f} = \mathcal{B} \mathcal{F}_1^{-1} \left(\frac{\lambda_c |R|}{|R|^5 + \lambda_c} \mathcal{F}_1 \hat{p} \right) \qquad \cdot 24$$

And it is seen that the filtering with $|R|$ in Fourier space which occurs when one calculates $\mathcal{R}^{-1}(\hat{p})$ is now replaced by filtering with

$$\frac{\lambda_c |R|}{|R|^5 + \lambda_c}$$

which is a band pass filter whose high frequency cut off is controlled by λ_c. The constant λ_c may be evaluated by substituting E·23 in E·22 giving

$$\int_0^{2\pi} d\Theta \int_{-\infty}^{\infty} dR \left| \frac{|R|^5}{|R|^5 + \lambda_c} \mathcal{F}_1 \hat{p} \right|^2 = e^2 \qquad \cdot 25$$

The comments of §·521 concerning changing $\le e^2$ to $= e^2$ are again relevant.

·523 Case 3 : $S(f) = ||f||^2_{B^2}$, $d(\hat{p},\mathcal{R}f) = ||\mathcal{F}^{-1}\mathcal{F}_1\hat{p} - \mathcal{F}_2 f||_{\mathbb{R}^2}$

The development again follows the same lines as §·521. Note the change in the constraint function d . In Cases 1 and 2 the constraint was that the Radon transform of the estimate $\hat{f}$ should be close to the observed projection data $\hat{p}$, now the constraint requires that their Fourier transforms are close or equivalently (by using Parseval's theorem) that their density functions are close.

From §·521

$$S(f) = ||f||^2_{B^2} = \int_0^{\pi} d\Theta \int_{-\infty}^{\infty} dR|R|\,|\mathcal{F}_2 f|^2$$

and by definition

$$d(\hat{p},\mathcal{R}f)^2 = \int_0^{\pi} d\Theta \int_{-\infty}^{\infty} dR|R|\,|\mathcal{F}_1\hat{p} - \mathcal{F}_2 f|^2 \ .$$

Applying the Calculus of variations to "min S subject to $d^2 \le e^2$" gives

$$|R|\mathcal{F}_2 f - \lambda_c |R|(\mathcal{F}_1\hat{p} - \mathcal{F}_2 f) = 0$$

hence

$$\mathcal{F}_2 f = \frac{\lambda_c}{1+\lambda_c}\,\mathcal{F}_1\hat{p} \quad \text{except possibly at} \quad R = 0$$

It is clear that since $\mathcal{F}_2\hat{f}$ is a constant multiple of $\mathcal{F}_1\hat{p}$ then $\hat{f}$ will be a constant multiple of $\mathcal{R}^{-1}(\hat{p})$ and the solution is trivial. This case is therefore taken no further.

·524 Case 4: $S(f) = ||\nabla^2 f||^2_{B^2}$, $d(\hat{p},\mathcal{R}f) = ||\mathcal{F}^{-1}\mathcal{F}_1\hat{p} - \mathcal{F}_2 f||_{\mathbb{R}^2}$

As in §·522

$$S(f) = 16\pi^4 \int_0^{\pi} d\Theta \int_{-\infty}^{\infty} dR|R|^5\,|\mathcal{F}_2 f|^2 \qquad \cdot 41$$

and by definition

$$d(\hat{p},\mathcal{R}f)^2 = \int_0^{\pi} d\Theta \int_{-\infty}^{\infty} dR\,|R|\,|\mathcal{F}_1\hat{p} - \mathcal{F}_2 f|^2 \qquad \cdot 42$$

Applying the Calculus of Variations to the problem "min S subject to $d^2 \le e^2$" gives

$$|R|^5 \mathcal{F}_2 f - \lambda_c \ |R| \ (\mathcal{F}_1 \hat{p} - \mathcal{F}_2 f) = 0$$

(Note that, without altering the problem, the factor $16\pi^4$ has been dropped from $S(f)$) and hence

$$\mathcal{F}_2 f = \frac{\lambda_c}{|R|^4 + \lambda_c} \ \mathcal{F}_1 \hat{p} \quad \text{except possibly at} \quad R = 0 \quad . \qquad \cdot 43$$

Thus (as in §·521)

$$\hat{f} = \mathcal{B} \mathcal{F}_1^{-1} \left(\frac{|R|\lambda_c}{|R|^4 + \lambda_c} \ \mathcal{F}_1 \hat{p} \right) \qquad \cdot 45$$

where λ_c is calculated by substituting E·43 into E·42 giving

$$\int_0^\pi d\Theta \int_{-\infty}^{\infty} dR \ |R| \ \left| \frac{|R|^4}{|R|^4 + \lambda_c} \ \mathcal{F}_1 \hat{p} \right|^2 = e^2 \qquad \cdot 46$$

The comments of §·521 concerning the effect of this filtering and the assumption of equality for the constraint are applicable here (with the obvious alterations in details).

·53 Implementation of modified reconstruction

·531 Discrete version of $\mathcal{B}\mathcal{F}_1^{-1}\left\{ \frac{\lambda_c \ R}{|R|^n + \lambda_c} \ \mathcal{F}_1 \hat{p} \right\}$: Cases 1 - 4

Consider first the function $\mathcal{F}_1^{-1}\left\{ \frac{\lambda_c \ |R|}{|R|^n + \lambda_c} \ \mathcal{F}_1 \hat{p} \right\}$ which was the result for each of the possibilities considered in §·52. This is the convolution of two functions $\mathcal{F}_1^{-1}\left\{ \frac{\lambda_c \ |R|}{|R|^n + \lambda_c} \right\}$ and $\hat{p}$ and it is required to calculate sample values of the convolution from a knowledge of sample values of $\hat{p}$. If $\hat{p}$ has been sampled at N_3 points spaced Δr apart then the sampled version of $\mathcal{F}_1^{-1}\{\ \}$ can be estimated as

$$(N_3\Delta r)^{-1}\ \underline{\underline{W}}^{-1}\ \ \lambda_d\ \ \underline{\underline{\Lambda}}(\underline{\underline{\Lambda}}^n + \lambda_d\underline{\underline{I}})^{-1}\underline{\underline{W}}\ \hat{\underline{p}} \qquad \cdot 1$$

where $\lambda_d = \lambda_c(N_3\Delta r)^n$ (see Appendix 2·4. Note that if $n = 1$, the condition "$S(R) \approx 0$ for all sufficiently large $|R|$" is not fulfilled. This possibility is discussed further in §·546, in conjunction with the question of choosing N_3).

Now consider the discrete implementation of $\mathcal{B}$. By definition

$$(\mathcal{B}\,t)(r,\phi) = \frac{1}{2}\int_0^{2\pi} t(r\cos(\phi - \theta),\theta)d\theta$$

The natural discrete version of this is

$$\Delta\phi \sum_{j=1}^{M} t(r\cos(\phi - \theta_j),\theta_j) \text{ where } \theta_j = (j-1)\pi/M$$

assuming $t(r,\phi) = t(-r,\phi + \pi)$, which is clearly the case for the functions under consideration here as this simply corresponds to reversing the positions of source and detector. However, this still assumes $t(r,\phi_j)$ considered as a function of r is defined on a continuum. Suppose now that $t(r,\phi_j)$ is known only at N_3 discrete points

$$r_k = \left\{k - \frac{N_3 - 1}{2}\right\}\ \Delta r \qquad k = 0(1)N_3 - 1$$

Define the vector $\underline{t}(\phi_j)$ (for each j) by

$$(\underline{t}(\phi_j))_k = t(r_k,\phi_j)$$

then $\underline{t}(\phi_j)$ may be interpolated by

$$\underline{v}\left(\frac{r}{\Delta r} + \frac{N_3 - 1}{2}\right)'\ \ \underline{t}(\phi_j)$$

where $\underline{v} = \underline{v}_1$ for step function interpolation or $\underline{v} = \underline{v}_2$ for linear interpolation (see Appendix 2·5). The function $(\mathcal{B}t)(r,\phi)$ is therefore approximated by

$$\Delta\phi \quad \sum_{j=1}^{M} \underline{v}(d_j)'\ \underline{t}(\phi_j) \qquad \cdot 2$$

where

$$d_j = \frac{r}{\Delta r}\cos(\phi - \phi_j) + \frac{N_3 - 1}{2} \quad . \qquad \cdot 3$$

Combining E·1 and E·2 gives:-

$$\hat{f}(r,\phi) = \frac{\Delta\phi}{N_3\Delta r} \sum_{j=1}^{M} \underline{v}(d_j)' \underline{\underline{W}}^{-1}(\lambda_d \underline{\underline{\Lambda}})(\underline{\underline{\Lambda}}^n + \lambda_d \underline{\underline{I}})^{-1} \underline{\underline{W}}\hat{\underline{p}}(\phi_j) \qquad \cdot 4$$

where d_j is defined by E·3 and $\underline{v} = \underline{v}_1$ or $\underline{v} = \underline{v}_2$ as appropriate.

·532 Estimation of λ_d

From §·52 E·19 and §·52 E·25 λ_c is given by

$$\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dR \left| \frac{|R|^n}{|R|^n + \lambda_c} \mathcal{F}_1 p \right|^2 = e_c^2 \qquad \cdot 5$$

(Note: e_c^2 (c for continuous) is now written rather than just e^2 as before in order to emphasise the distinction between this and e_d^2 (d for discrete) which is shortly to be introduced.)

Approximating both integrals gives

$$2\Delta\phi\Delta R \sum_{j=1}^{M} \sum_{k=-\frac{N_3}{2}+1}^{N_3/2} \left| \frac{|k|^n \Delta R^n}{|k|^n \Delta R^n + \lambda_c} (\mathcal{F}_1\hat{p})(k\Delta R, \phi_j) \right|^2 = e_c^2$$

and applying Appendix 1·1·3 E·6 to $(\mathcal{F}_1\hat{p})(k\Delta R,\phi_j)$ where $\Delta R = (N_3\Delta r)^{-1}$, and substituting for $|k|^n$ in terms of $\underline{\underline{\Lambda}}$ as in §·531 and Appendix 2·4 gives

$$2\Delta\phi\Delta R \sum_{j=1}^{M} \sum_{k=0}^{N_3-1} \left| \frac{\Lambda_k^n}{\Lambda_k^n + \lambda_d} \Delta r \sqrt{N_3} (\underline{\underline{W}}\hat{\underline{p}})_k(\phi_j) \right|^2 = e_c^2 \qquad \cdot 6$$

but $2\Delta\phi\Delta R\, \Delta r^2 N_3 = 2\Delta\phi\Delta r$ hence λ_d may be estimated from

$$\sum_j [\underline{\underline{\Lambda}}^n(\underline{\underline{\Lambda}}^n + \lambda_d\underline{\underline{I}})^{-1}\underline{\underline{W}}\hat{\underline{p}}]^T [\underline{\underline{\Lambda}}^n(\underline{\underline{\Lambda}}^n + \lambda_d\underline{\underline{I}})^{-1}\underline{\underline{W}}\hat{\underline{p}}] = e_d^2 \qquad \cdot 7$$

where $\lambda_d = \lambda_c(N_3\Delta r)^n$ and $e_d^2 = e_c^2/2\Delta\phi\Delta r$.

·533 Estimation of e_d^2

·5331 Cases 1,2.

The original constraint equation is

$$d(\hat{p},\mathcal{R}f)^2 \le e_c^2$$

a value is therefore assigned to e_c^2 using

$$e_c^2 = E\{d(\hat{p},E\{\hat{p}\})^2\} \qquad \cdot 75$$

so that the distance between $\hat{p}$ and $\mathcal{R}f$ is constrained to be no more than the most probable distance between the particular sample (estimate) of $\hat{p}$ in use on this occasion and its mean value. Note that this is not a derived relation but merely the authors personal choice, others may choose differently. As it stands E·75 cannot be evaluated as $\hat{p}$ is estimated only at discrete points a discrete analogue of d is therefore considered.

By definition for Cases 1 and 2:-

$$\begin{aligned} d(p_1,p_2)^2 &= ||p_1 - p_2||^2_{B^1 \times S^1} \\ &= \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dr\, |p_1 - p_2|^2 \\ &\simeq 2\Delta\phi\Delta r \sum_{j=1}^{M} \sum_{k=0}^{N_3-1} |(\underline{p}_1)_k(\phi_j) - (\underline{p}_2)_k(\phi_j)|^2 \\ &= 2\Delta\phi\Delta r \sum_j (\underline{p}_1 - \underline{p}_2)^T(\underline{p}_1 - \underline{p}_2) \end{aligned}$$

Substituting $\hat{\underline{p}}$ for $\underline{p}_1$ and $E\{\hat{\underline{p}}\}$ for $\underline{p}_2$, e_c^2 is evaluated as

$$e_c^2 = E\{2\Delta\phi\Delta r \sum_j (\hat{\underline{p}} - E\{\hat{\underline{p}}\})^T(\hat{\underline{p}} - E\{\hat{\underline{p}}\})\}$$

whence

$$e_d^2 = \frac{e_c^2}{2\Delta\phi\Delta r} = \sum_j \mathrm{tr}\, \underline{\underline{V}}\,(\hat{\underline{p}}(\phi_j)) \qquad \cdot 8$$

where $\underline{\underline{V}}(\hat{\underline{p}}(\phi_j))$ is the covariance matrix of $\hat{\underline{p}}(\phi_j)$.

·5332 Case 4

In this case one cannot produce a line of reasoning similar to that used to obtain E·8, it is straightforward to see what goes amiss.

As above

$$d(\hat{p},\mathcal{R}f)^2 \le e_c^2$$

and e_c^2 evaluated as

$$e_c^2 = E\{d(\hat{p},E\{\hat{p}\})\}.$$

Following the reasoning of §·5331

$$\begin{aligned} d(p_1,p_2)^2 &= \|\mathcal{F}_1 p_1 - \mathcal{F}_1 p_2\|^2_{\mathbb{R}^2} \\ &= \int_0^\pi d\phi \int_{-\infty}^{\infty} dR\ |R|\, |\mathcal{F}_1 p_1 - \mathcal{F}_1 p_2|^2 \\ &\simeq \Delta\phi\Delta R \sum_{j=1}^{M} \sum_{k=-\frac{N_3}{2}+1}^{N_3/2} |k|\ \Delta R |\mathcal{F}_1 p_1(k\Delta R,\phi_j) - \mathcal{F}_1 p_2(k\Delta R,\phi_j)|^2 \\ &= \Delta\phi\Delta R \sum_{j=1}^{M} \sum_{k=0}^{N_3-1} \Delta_k \Delta R \left| \Delta r \sqrt{N_3} \left((\underline{\underline{W}}\ \hat{\underline{p}}_1)_k(\phi_j) - (\underline{\underline{W}}\ \hat{p}_2)_k(\phi_j) \right) \right|^2 \end{aligned}$$

and since $\Delta\phi\Delta R^2 \Delta r^2 N_3 = \Delta\phi/N_3$

$$d(p_1,p_2)^2 = \frac{\Delta\phi}{N_3} \sum_j (\underline{p}_1 - \underline{p}_2)^T \underline{\underline{W}}^T \underline{\underline{\Delta}}\ \underline{\underline{W}}(\underline{p}_1 - \underline{p}_2)$$

so that

$$e_d^2 = \frac{e_c^2}{2\Delta\phi\Delta r} = \frac{1}{2\Delta r N_3} E\{\sum_j (\hat{\underline{p}} - E\{\hat{\underline{p}}\})^T \underline{\underline{W}}^T\ \underline{\underline{\Delta}}\ \underline{\underline{W}}(\hat{\underline{p}} - E\{\hat{\underline{p}}\})\}.$$

This expression cannot be evaluated without knowing both $\hat{\underline{p}}$ and $E\{\hat{\underline{p}}\}$ and a value for $E\{\hat{\underline{p}}\}$ is not available. Note that if this information was available it would still be necessary to check that $\mathcal{F}_1(p_1 - p_2)$ was negligible for $|R| > N_3\Delta R/2$ in order to justify the change from

$$\int dR \quad \text{to} \quad \sum_{k=-N_3/2+1}^{N_3/2} .$$

·54 Approximation of S(R) by $\underline{S}$

·541 General discussion : n > 2

In §·52 it was seen that $\hat{f}$ may be estimated by

$$\hat{f} = \mathcal{B}\,\mathcal{F}_1^{-1}\left\{ \frac{\lambda_c\,|R|}{|R|^n + \lambda_c}\,\mathcal{F}_1\hat{p} \right\}$$

and in §·531 that for practical purposes $\mathcal{F}_1^{-1}\{\ldots\}$ must be implemented as

$$(N_3\Delta r)^{-1}\underline{\underline{W}}^{-1}\lambda_d\,\underline{\underline{\Lambda}}\,(\underline{\underline{\Lambda}}^n + \lambda_d\underline{\underline{I}})^{-1}\underline{\underline{W}}\,\hat{\underline{p}}\ ,$$

this section is devoted to considering just how well such a discrete implementation approximates the continuous case.

Define the functions s and S by

$$S(R) = \frac{\lambda_c\,|R|}{|R|^n + \lambda_c}$$

$$s = \mathcal{F}_1^{-1}S$$

and consider some of their properties. The function S is illustrated in fig. 2·5, for $n > 1$ $S \in \mathcal{L}^2(\mathbb{R})$, has unbounded support and is not differentiable at $R = 0$. The function s is defined and in $\mathcal{L}^2$ (since $S \in \mathcal{L}^2$) and also has unbounded support (else $s \in \mathcal{L}^2 \Rightarrow rs \in \mathcal{L}^2 \Rightarrow S'$ exists $\forall R$ (see Bochner et al, 1949 : p 126 Th. 62) $\Rightarrow S'(0)$ exists which is not true).

In order to pass from the continuous to the discrete equation in §·53 it was necessary to invoke the results of Appendices 2·1 and 2·4 and these require that both S and s are approximately zero for all sufficiently large R and r. Since it has just been seen that s and S both have unbounded support it is natural to examine the conditions under which s and S are sufficiently small.

Now compare the parallel parts of the continuous and discrete version of §·531 E·1.

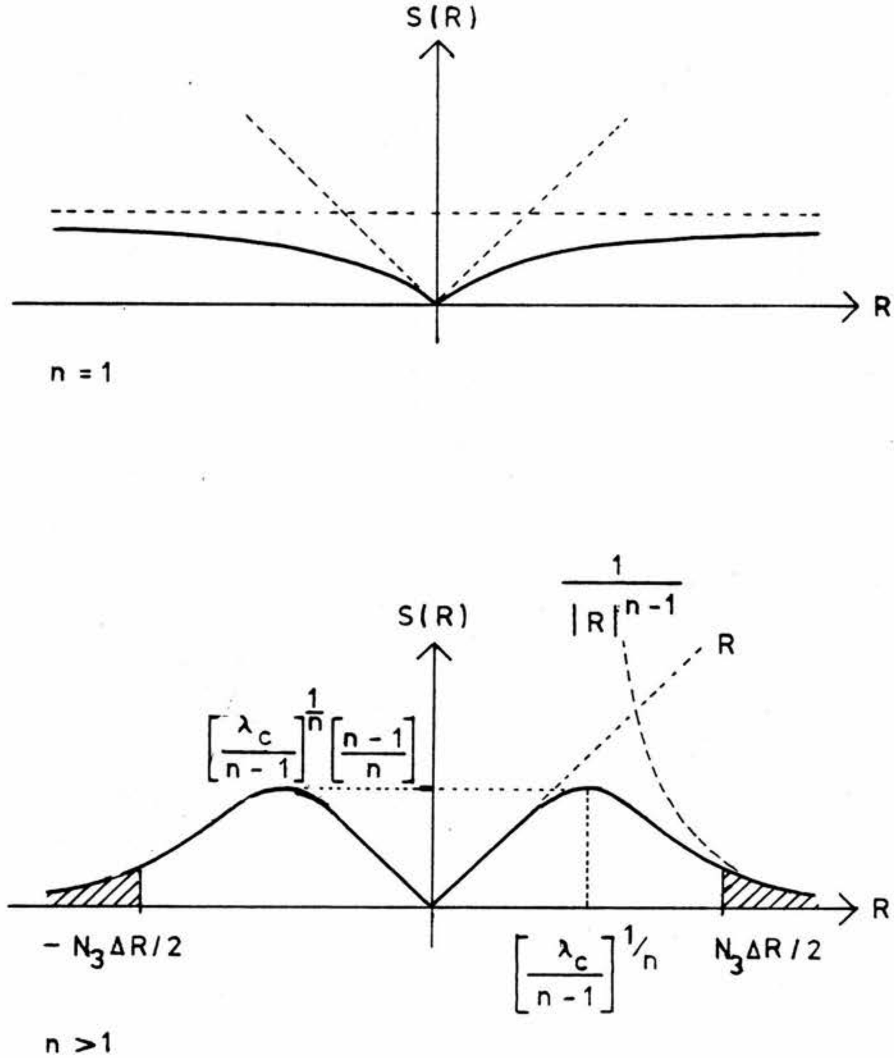


fig. 2·5 : Graph of S(R).

The function $S(R) = \dfrac{\lambda_c \, |R|}{|R|^n + \lambda_c}$ is replaced by

$$\Delta R \; \lambda_d \; \underline{\underline{\Lambda}} (\underline{\underline{\Lambda}}^n + \lambda_d \underline{\underline{I}})^{-1} ,$$

the transform $s = \mathscr{F}_1^{-1} S$ is replaced by

$$\Delta R \; \underline{\underline{W}}^{-1} \lambda_d \; \underline{\underline{\Lambda}} (\underline{\underline{\Lambda}}^n + \lambda_d \underline{\underline{I}})^{-1} \underline{\underline{W}} ,$$

and the continuous convolution $s * \hat{p}$ is replaced by

$$\Delta R \; \underline{\underline{W}}^{-1} \lambda_d \; \underline{\underline{\Lambda}} (\underline{\underline{\Lambda}}^n + \lambda_d \underline{\underline{I}})^{-1} \underline{\underline{W}} \; \hat{\underline{p}} .$$

There are three approximations implicit in these changes and these are now considered in detail:-

1) the integration implicit in the transform $s = \mathscr{F}_1^{-1} S$ is changed to the relation $\underline{s} = \underline{\underline{W}}^{-1} \underline{S}$, and as seen below this is equivalent to substituting numerical integration. If Δr is the sample spacing and ΔR given by $N_3 \Delta r \Delta R = 1$ then

$$s(j\Delta r) = \int_{-\infty}^{\infty} \frac{\lambda_c |R|}{|R|^n + \lambda_c} \; e^{2\pi i (j\Delta r) R} \, dR$$

$$\simeq \left\{ \begin{array}{l} \displaystyle \Delta R \sum_{k=-\frac{N_3}{2}+1}^{\frac{N_3}{2}-1} \frac{\lambda_c |k\Delta R|}{|k\Delta R|^n + \lambda_c} \; e^{2\pi i jk/N_3} \\[2ex] \displaystyle + \frac{1}{2} \Delta R \; \frac{\lambda_c |k\Delta R|}{|k\Delta R|^n + \lambda_c} \; e^{2\pi i jk/N_3} \Bigg|_{k = -\frac{N_3}{2}} \\[2ex] \displaystyle + \frac{1}{2} \Delta R \; \frac{\lambda_c |k\Delta R|}{|k\Delta R|^n + \lambda_c} \; e^{2\pi i jk/N_3} \Bigg|_{k = \frac{N_3}{2}} \end{array} \right. .$$

$$= \begin{cases} \Delta R^2 \sum_{k=0}^{\frac{N_3}{2}-1} \dfrac{\lambda_d |k|}{|k|^n + \lambda_d} e^{2\pi ijk/N_3} \\ + \Delta R^2 \dfrac{\lambda_d |k|}{|k|^n + \lambda_d} e^{2\pi ijk/N_3} \Bigg|_{k = \frac{N_3}{2}} \\ + \Delta R^2 \sum_{k=\frac{N_3}{2}+1}^{N_3-1} \dfrac{\lambda_d |k - N_3|}{|k - N_3|^n + \lambda_d} e^{2\pi ijk/N_3} \end{cases}$$

where $\lambda_d = \lambda_c \Delta R^{-n}$

$$= \Delta R^2 \sum_{k=0}^{N_3-1} \frac{\lambda_d \Lambda_k}{\Lambda_k^n + \lambda_d} e^{2\pi ijk/N_3}$$

and this (except for a multiplicative constant) is $\underline{\underline{W}}^{-1}\underline{s}$. This in itself contains two approximations:-

(a) The limits $\int_{-\infty}^{\infty}$ are truncated to $\int_{-N_3\Delta R/2}^{N_3\Delta R/2}$, so that the two tails $|R| > N_3\Delta R/2$ are neglected.

(b) Integration over this now finite range is approximated by the trapezium rule, i.e. the interval $[-N_3\Delta R/2, N_3\Delta R/2]$ is divided into N_3 equal sections of length ΔR and it is assumed that the function takes its average value in each section.

2) Having thus estimated $s(r)$ for $r = j\Delta r$, the infinite linear convolution

$$(s * \hat{p})(r_j) \simeq \Delta r \sum_{k=-\infty}^{\infty} s\Big((j - k)\Delta r\Big)\hat{p}(r_k)$$

where $r_j = \left(j - \dfrac{N_3-1}{2}\right)\Delta r$, is replaced by the circular convolution

$$(s * \hat{p})(r_j) \simeq \sum_{k=0}^{N_3-1} s^*_{jk} \hat{p}(r_k)$$

where $\underline{\underline{s}}^*$ is defined in the obvious fashion (see appendices 2·1 and 2·4).

In §§·542 - ·545 each of these approximations is considered in turn. This leads to restrictions on the value of λ (which are taken further in §3) and also to a criterion for assigning N_3 (in §·545). The whole of this discussion applies only to the case $n > 2$, and is frequently particularised to the case $n = 5$. The other case of interest, $n = 1$, is considered separately in §·546.

·542 Restrictions on λ considered analytically.

·5421 Maximum value of λ_c

The requirement that the error (1a) of §·541 is kept acceptably small is considered and results in a condition relating Δr and λ_c. The most obvious way of ensuring that truncating the tails $|R| > N_3\Delta R/2$ causes little error is to require that $(2\Delta r)^{-1} = N_3\Delta R/2$ is chosen sufficiently large for the area under the truncated tail of $S(R)$ to be negligible compared with the area under the main hump. In terms of integrals this means choosing Δr such that

$$\int_{(2\Delta r)^{-1}}^{\infty} \frac{\lambda_c |R|}{|R|^n + \lambda_c}\, dR \le \theta \int_0^{\infty} \frac{\lambda_c |R|}{|R|^n + \lambda_c}\, dR \qquad \cdot 1$$

where $\theta \in [0,1]$. With suitable substitution this becomes

$$\int_{(2\Delta r\lambda_c^{1/n})^{-1}}^{\infty} \frac{\zeta}{\zeta^n + 1}\, d\zeta \le \theta \int_0^{\infty} \frac{\zeta}{\zeta^n + 1}\, d\zeta$$

Now
$$\int_{(2\Delta r\lambda_c^{1/n})^{-1}}^{\infty} \frac{\zeta}{\zeta^n + 1}\, d\zeta = \int_{(2\Delta r\lambda_c^{1/n})^{-1}}^{\infty} \frac{\zeta^{1-n}}{1 + \zeta^{-n}}\, d\zeta$$
$$\le \int_{(2\Delta r\lambda_c^{1/n})^{-1}}^{\infty} \zeta^{1-n}\, d\zeta$$
$$= \frac{(2\Delta r\lambda_c^{1/n})^{n-2}}{n-2} \qquad n \neq 2$$

and it is shown in Appendix 2·6 that

$$A = \int_0^{\infty} \frac{\zeta}{\zeta^5 + 1}\, d\zeta = \frac{32\pi}{25} \sin^3\frac{\pi}{5} \cos\frac{\pi}{5} \simeq 0\cdot 6607$$

Hence a sufficient condition for E·1 to hold (in the case n = 5) is

$$\frac{1}{3}\,(2\Delta r\lambda_c^{1/5})^3 \le \theta A \quad \text{or by rearranging}$$

$$\Delta r \le \frac{(3\theta A)^{1/3}}{2\lambda_c^{1/5}}$$

If the substitution $\lambda_c = \lambda_d\,\Delta R^5$ is made then this becomes

$$N_3 \ge \frac{2\lambda_d^{1/5}}{(3\theta A)^{1/3}} \qquad \cdot 3$$

and this condition can be easily monitored in any computer implementation of the results. One might expect to use values of θ of about 0·01 in which case $(3\theta A)^{1/3} \simeq 0{\cdot}27$.

It is not clear from this whether the basic result is E·2 or E·3. The first suggests a necessary condition on the constants Δr and λ_c, while the second suggests that N_3 must be chosen sufficiently large for the inequality to hold. However, from §·5331 it can be seen that e_c^2 is determined by the original body section f, the number of counts per cell ℓ, the collimator radius a, and the number of data points N (N.B. by N not N_2 or N_3). That is, once the scanner parameters $a, \Delta r, \ell$ etc. (see §3·1) have been chosen, e_c^2 is already determined (although unknown). Its value is thus independent of any constants which are chosen during the data processing and, in particular, independent of N_3. Since λ_c is determined by §·532 E·5 in terms of f, e_c^2 and n it is also independent of N_3. The basic equation is therefore E·2 which states requirements on Δr and λ_c and E·3 simply provides an easy method of checking the requirement when actually performing data processing. It follows that E·3 is therefore true for all N_3 or for none, a fact which is not surprising when it is remembered that $\lambda_d^{1/5} = \lambda_c^{1/5}\,\Delta r N_3$ and is thus dependent on N_3.

·5422 Minimum value of λ_d

The requirement that the error (1b) of §·541 is kept acceptably small is now considered and results in a minimum value for λ_d. The error involved in using the trapezium rule for numerical integration decreases with the square of ΔR (Noble, 1964: Vol II §9·2 p 230) and it is therefore necessary to choose ΔR sufficiently small for $S(R)$ to be well represented.

For n = 5, straightforward calculation shows that $S(R)$ at $R = 2\lambda_c^{1/5}$ has a value which is about 10% of its peak value,

$$S\left(\left(\frac{\lambda_c}{n-1}\right)^{1/n}\right) = \frac{n-1}{n}\left(\frac{\lambda_c}{n-1}\right)^{1/n}$$

and the requirement that $S(R)$ is sampled sufficiently often is defined to mean that there must be at least N_4 samples of $S(R)$ over this peak (the peak being taken as $R \in [0, 2\lambda_c^{1/5}]$) i.e. that

$$N_4 \Delta R \leq 2\lambda_c^{1/5}$$

or $$\Delta R \leq \frac{2\lambda_c^{1/5}}{N_4} \qquad \cdot 4$$

and since $\lambda_c^{1/5} = \lambda_d^{1/5}\,\Delta R$ this may be written

$$\lambda_d \geq \left(\frac{N_4}{2}\right)^5 \qquad \cdot 5$$

A condition which can easily be checked in any computer program to ensure that $S(R)$ is sufficiently finely sampled.

Substituting $\Delta R = (N_3 \Delta r)^{-1}$ in E·4 gives

$$N_3 \Delta r \geq \frac{N_4}{2\lambda_c^{1/5}}$$

and since $\Delta r, N_4$ and λ_c (see §·5421) are independent of N_3, the condition may always be satisfied by choosing N_3 sufficiently large. Thus the requirement is not a condition on Δr but on N_3.

As an example, choosing $N_4 = 10$ gives the constraint $\lambda_d \geq 3125$.

·543 Restrictions on λ considered numerically

The use of the constraints E·3 and E·5 of §·542 requires the assignment of values to θ and N_4 and it is not at all obvious just what these values should be. (The tentative values $\theta = 0{\cdot}01$ and $N_4 = 10$ used in §·542 being simply illustrative.) The reason for this is that θ and N_4 are defined in terms of their numerical effect on the Fourier space quantity S , whereas the quantity of interest is the real space quantity $s * \hat{p}$. The analytic approach of §·542 was therefore included largely to introduce the ideas involved and is now dropped in favour of a numerical approach which utilises two different constants (θ_2 and m) which are defined in terms of the effect of the approximations on the real space quantity s.

The account of this numerical approach is divided into three parts. In this section, the definitions of θ_2 and m are given together with a brief summary of the new conditions which replace those of E·3 and E·5 in §·542. In Appendix 2·7 the theory developing the definitions of θ_2 and m is given, and in §2·8· a complete set of software listings and outputs is given.

·5431 Definitions of θ_2 and m

The constraint E·3 that

$$\lambda_d \le \lambda_{d\ max} = \left(\frac{1}{2}\,(3\theta A)^{1/3} N_3\right)^5$$

of §·542 was derived from the requirement that $S(R)$ was sampled for sufficiently large R. This is equivalent to the real space requirement that $s(r)$ is sampled sufficiently finely. Now the function $s(r) = \mathcal{F}^{-1} S(R)$ can be evaluated analytically by using the partial fraction expansion

$$\frac{x}{x^5 + 1} = -\frac{1}{5} \sum_{k=1}^{5} \frac{\omega^{2k}}{x + \omega^k}$$

(where ω is the primitive 5th root of unity) and then transforming each individual term (Erdelyi et al, 1954: §1·2(7),§2·2(10)) giving an expression for $s(r)$ in terms of sine and cosine integrals, which suggests that $s(r)$ has the form of a damped oscillation. The requirement for sampling S for large enough R, is thus equivalent to sampling s finely enough for each half cycle of s to be properly reproduced.

The constraint E·5 that

$$\lambda_d \geq \lambda_{d\ min} = \left(\frac{N_4}{2}\right)^5$$

given in §·542 arose from the requirement that $S(R)$ be sufficiently finely sampled to be properly represented by its sampled values, and this is equivalent to requiring that $s(r)$ be sampled for large enough r to avoid aliasing in real space.

Let $\underline{S}$ denote the sampled values of S (as in Appendix 2·4) and let $\underline{s} = \underline{\underline{W}}^{-1}\underline{S}$ be its D.F.T., $\lambda_{d\ min}$ and $\lambda_{d\ max}$ are now defined in terms of their effect on the real space vector $\underline{s}$ by:-

> $\lambda_{d\ min}$ is such that $\lambda_d \geq \lambda_{d\ min}$ implies that the error in any component of $\underline{s}$ due to aliasing is less than a given small fraction θ_2 of the largest component of $\underline{s}$ ·1

and

> $\lambda_{d\ max}$ is such that $\lambda_d \leq \lambda_{d\ max}$ implies that $\underline{s}$ is a sufficiently finely sampled version of s for there to be at least m components of $\underline{s}$ corresponding to each half cycle of s. ·2

Reasoning from these definitions (Appendix 2·7) and using computer programs (Appendix 2·8) one arrives at the facts that:-

> $\lambda_{d\ min}$ can be graphed as a function of θ_2 (expressed as a percentage) as in fig 2·6, the values for which are tabulated as in fig 2·7. For practical purposes, values of $\lambda_{d\ min}$ at points other than those tabulated are taken as the linearly interpolated values on the $\log(\lambda_{d\ min})/\log(\theta_2)$ graph. ·3

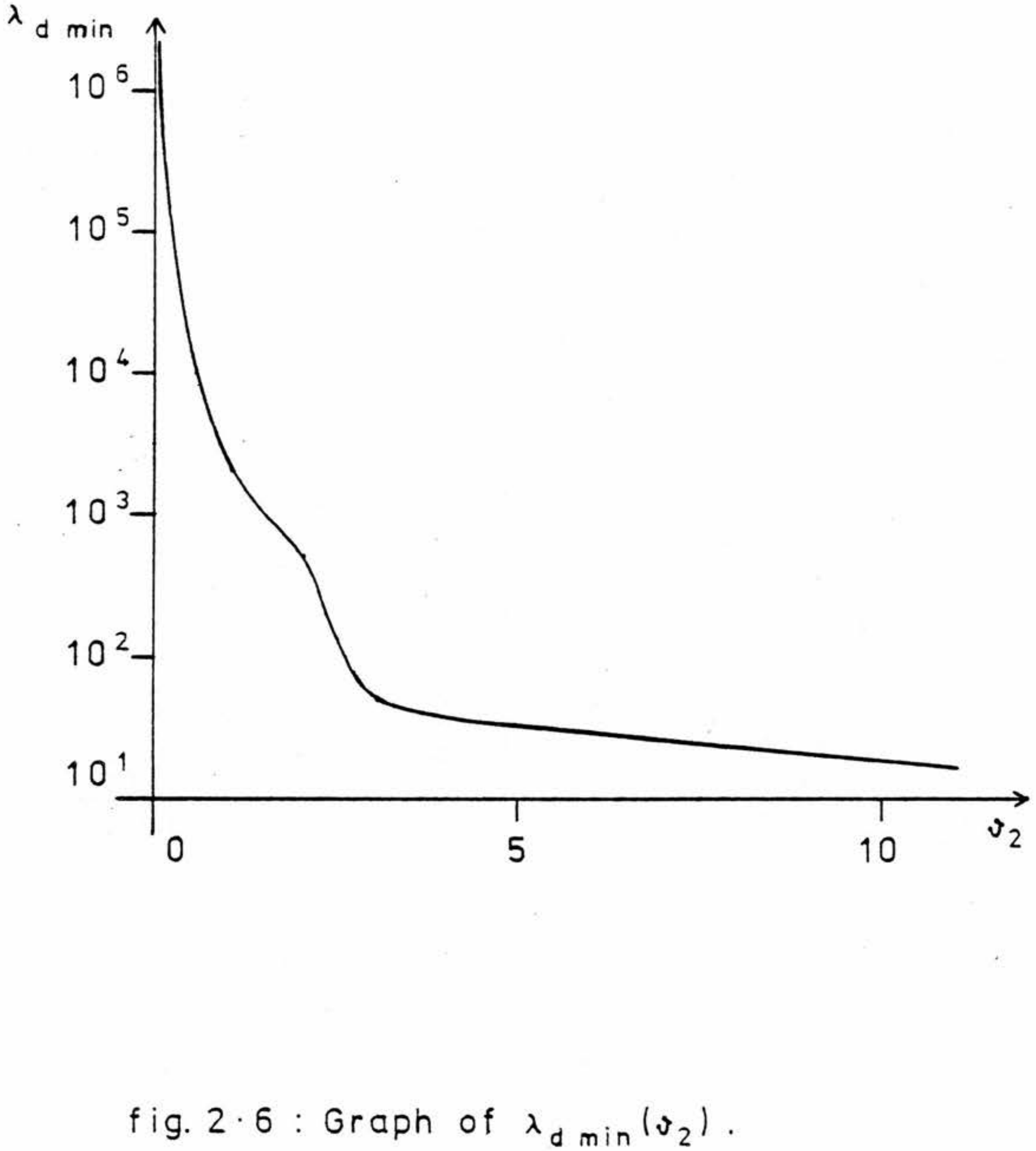


fig. 2·6 : Graph of $\lambda_{d\ min}(\vartheta_2)$.

TABULATED VALUES FOR LAMDA min AND ERROR

LAMDA min	ERR	LOG (LAMDA min)	LOG (ERR)
1.0000000000@ -1	6.2301135229@ 1	-2.3025850930@ 0	4.1319796476@ 0
2.0000000000@ -1	4.7244438212@ 1	-1.6094379124@ 0	3.8553349372@ 0
5.0000000000@ -1	3.7541369346@ 1	-6.9314718056@ -1	3.6254435075@ 0
1.0000000000@ 0	3.7272222475@ 1	0.0000000000@-99	3.6182483434@ 0
2.0000000000@ 0	3.5465183599@ 1	6.9314718056@ -1	3.5685514715@ 0
5.0000000000@ 0	2.6176862434@ 1	1.6094379124@ 0	3.2648759074@ 0
1.0000000000@ 1	1.7180774884@ 1	2.3025850930@ 0	2.8437910194@ 0
2.0000000000@ 1	9.2107879916@ 0	2.9957322736@ 0	2.2203754049@ 0
5.0000000000@ 1	3.0724623521@ 0	3.9120230054@ 0	1.1224793092@ 0
1.0000000000@ 2	2.6576378558@ 0	4.6051701860@ 0	9.7743770413@ -1
2.0000000000@ 2	2.3636362043@ 0	5.2983173665@ 0	8.6020119780@ -1
5.0000000000@ 2	2.0606166784@ 0	6.2146080984@ 0	7.2300529646@ -1
1.0000000000@ 3	1.5323562225@ 0	6.9077552790@ 0	4.2680656549@ -1
2.0000000000@ 3	1.0629084570@ 0	7.6009024595@ 0	6.1008978066@ -2
5.0000000000@ 3	7.4199984496@ -1	8.5171931914@ 0	-2.9840624476@ -1
1.0000000000@ 4	5.6667635482@ -1	9.2103403720@ 0	-5.6796694100@ -1
2.0000000000@ 4	4.2800271746@ -1	9.9034875525@ 0	-8.4862573421@ -1
5.0000000000@ 4	2.9685618982@ -1	1.0819778284@ 1	-1.2145074668@ 0
1.0000000000@ 5	2.2497575697@ -1	1.1512925465@ 1	-1.4917626294@ 0
2.0000000000@ 5	1.7032097557@ -1	1.2206072646@ 1	-1.7688969659@ 0
5.0000000000@ 5	1.1822442711@ -1	1.3122363377@ 1	-2.1351705362@ 0
1.0000000000@ 6	9.0842932454@ -2	1.3815510558@ 1	-2.3986232807@ 0

fig. 2·7

and

$\lambda_{d\ max}$ can be expressed by the formula

$$\lambda_{d\ max} = 2{\cdot}1877 \times 10^{-3} \left(\frac{N_3}{m-1}\right)^5 \quad . \qquad \cdot 4$$

These two results are used as the definitions of $\lambda_{d\ min}$ and $\lambda_{d\ max}$ in all future discussion unless otherwise stated.

·544 Support of s(r)

The concept of the support of an arbitary function f on $\mathbb{R}$ is well defined. This section develops a rough generalisation of this concept for the discrete function $\underline{s}^*$ and this generalisation is utilised in §·545 when considering the effects of the approximation (2) of §·541.

It is straightforward to show that increasing the value of λ_d in $\underline{S}^*$ is roughly equivalent to expanding the graph of $\underline{S}^*$ along the abscissa (followed by multiplication by a constant), and it is to be expected that this will result in increasing the number of components of $\underline{s}^*$ which are approximately zero, and this has obvious implications when calculating $\underline{\underline{s}}^* \hat{\underline{p}}$. (This discussion is most simply appreciated by considering first the quantities λ_c, S(R) and s(r)). It is therefore of interest to arrive at an expression for the number of adjacent zero components in $\underline{s}^*$ (there may also be further isolated zeros which are of no interest in the present discussion).

It is a standard theorem of Fourier analysis that

$$F(R) = \mathcal{F}_1\{f(r)\} \Leftrightarrow F(R/K) = \mathcal{F}_1\{Kf(Kr)\},$$

unfortunately there is no discrete counterpart of this. For this reason, when finding an expression for the number of zeros in $\underline{s}^*$ in terms of λ_d, it is necessary to use the relations between $\underline{S}^*$ and S, S and s, and s and $\underline{s}^*$.

Let f be some unspecified function. The various relationships between $\underline{F}^*$, F, f and $\underline{f}^*$ are illustrated diagrammatically in fig 2·8. The relationship between $\underline{f}^*$ and f is given in Appendix 2·1 as

$$(\underline{f}^*)_k = \begin{cases} \Delta r\ f(k\Delta r) & 0 \le k \le N_3/2 \\ \Delta r\ f((k - N_3)\Delta r) & N_3/2 \le k \le N_3 - 1 \end{cases} \qquad \cdot 1$$

$$\text{provided} \quad f(r) \simeq 0 \quad \forall |r| \ \ge N_3\Delta r/2. \qquad \cdot 15$$

A similar set of conditions is illustrated for the function $K\,F(R/K)$ and its transform $K^2 f(Kr)$. Defining $\underline{f}^{K*}$ to be Δr times the sampled version of $K^2 f(Kr)$ then

$$(\underline{f}^{K*})_k = \begin{cases} K^2\Delta r\ f(kK\Delta r) & 0 \le k \le N_3/2 \\ K^2\Delta r\ f((k - N_3)K\Delta r) & N_3/2 \le k \le N_3 - 1 \end{cases} \qquad \cdot 2$$

$$\text{provided} \quad K^2 f(Kr) \simeq 0 \quad \forall |r| \ge \ K\,N_3\Delta r/2 \ . \qquad \cdot 25$$

Assume that f is such that condition E·15 is satisfied and that $K > 1$, then E·25 is also satisfied. From E·15 and E·2

$$\left.\begin{aligned} (\underline{f}^{K*})_k \simeq 0 \quad &\text{if} \quad |k\ K\Delta r| \ge \frac{N_3\Delta r}{2} \\ &\Leftrightarrow \quad k \ge K^{-1}N_3/2 \end{aligned}\right\} \text{ for } \ 0 \le k \le N_3/2$$

and

$$\left.\begin{aligned} (\underline{f}^{K*})_k \simeq 0 \quad \text{if} \quad &|(k - N_3)K\Delta r| \ge \frac{N_3\Delta r}{2} \\ &\Leftrightarrow \quad k \le (1 - \frac{K^{-1}}{2})\ N_3 \end{aligned}\right\} \text{ for } \ N_3/2 \le k \le N_3 - 1.$$

Hence

$$(\underline{f}^{K*})_k \simeq 0 \quad \text{for} \quad \frac{K^{-1}}{2} N_3 \le k \le N_3 - \frac{K^{-1}}{2} N_3$$

$$\text{i.e. for} \quad \frac{N_3}{2} - (1 - K^{-1})\ \frac{N_3}{2} \le k \le \frac{N_3}{2} + (1 - K^{-1})\ \frac{N_3}{2}$$

and $\underline{f}^{K*}$ has at least $2 \times (1 - K^{-1})N_3/2 = (1 - K^{-1})N_3$ components which are adjacent and approximately zero. (There may of course be other isolated zeros.)

$$\begin{tikzcd}[column sep=huge, row sep=large]
F(R) \arrow[r, "\mathcal{F}^{-1}"] \arrow[d, "N_3 \text{ samples with spacing } \Delta R"] & f(r) \arrow[d, "N_3 \text{ samples with spacing } \Delta r"] \\
\underline{F}^* \arrow[r, "\underline{\underline{W}}^{-1}"] & \underline{f}^*
\end{tikzcd}$$

where
$$(\underline{f}^*)_k = \begin{cases} \Delta r \ f(k\,\Delta r) & 0 \leqslant k \leqslant N_3/2 \\ \Delta r \ f([k-N_3]\,\Delta r) & N_3/2 \leqslant k \leqslant N_3 - 1 \end{cases}$$

$$(\underline{F}^*)_k = \begin{cases} N_3^{-1/2} \ F(k\,\Delta R) & 0 \leqslant k \leqslant N_3/2 \\ N_3^{-1/2} \ F([k-N_3]\,\Delta R) & N_3/2 \leqslant k \leqslant N_3 - 1 \end{cases}$$

$$\begin{tikzcd}[column sep=huge, row sep=large]
K F(R/K) \arrow[r, "\mathcal{F}^{-1}"] \arrow[d, "N_3 \text{ samples with spacing } \Delta R"] & K^2 f(Kr) \arrow[d, "N_3 \text{ samples with spacing } \Delta r"] \\
\underline{F}^{K*} \arrow[r, "\underline{\underline{W}}^{-1}"] & \underline{f}^{K*}
\end{tikzcd}$$

where
$$(\underline{f}^{K*})_k = \begin{cases} K^2 \Delta r \ f(kK\Delta r) & 0 \leqslant k \leqslant N_3/2 \\ K^2 \Delta r \ f([k-N_3]K\Delta r) & N_3/2 \leqslant k \leqslant N_3 - 1 \end{cases}$$

$$(\underline{F}^{K*})_k = \begin{cases} N_3^{-1/2} \ K\,F(k\,\Delta R/K) & 0 \leqslant k \leqslant N_3/2 \\ N_3^{-1/2} \ K\,F([k-N\ ]\,\Delta R/K) & N_3/2 \leqslant k \leqslant N_3 - 1 \end{cases}$$

fig. 2·8 : Relation between $\underline{F}^*$, F, f and $\underline{f}^*$.

This is now related to the function $S(R)$. Let

$$S_{min}(R) = \frac{\lambda_{c\ min}|R|}{|R|^5 + \lambda_{c\ min}}$$

where $\lambda_{c\ min} = \lambda_{d\ min}\ \Delta R^5$, and let

$$S(R) = \frac{\lambda_c\ |R|}{|R|^5 + \lambda_c}$$

where $\lambda_{d\ min} \le \lambda_c \Delta R^{-5} \le \lambda_{d\ max}$, then

$$S(R) = K\ S_{min}(R/K)$$

where $K = \left(\frac{\lambda_c}{\lambda_{c\ min}}\right)^{1/5} \ge 1.$

Now put $F = S_{min}$ so that $K\ F(R/K) = S(R)$. Then

$$\underline{F}^* = \underline{S}^*_{min} \quad \text{and} \quad \underline{F}^{K*} = \underline{S}^*$$

where $\underline{S}^*_{min}$ is $\underline{S}^*$ with $\lambda_d = \lambda_{d\ min}$. By definition of $\lambda_{d\ min}$ in §·5431 E·1 $s_{min}(r)$ is negligible for $|r| > N_3\Delta r/2$ (to within the accuracy set by θ_2), hence $f = s_{min}$ is negligible for $|r| > N_3\Delta r/2$. It follows immediately that

$$(\underline{f}^{K*})_k = (\underline{s}^*)_k \simeq 0 \quad \text{for} \quad \frac{N_3}{2} - (1 - K^{-1})\ \frac{N_3}{2} \le k \le \frac{N_3}{2} + (1 - K^{-1})\ \frac{N_3}{2}$$

and that $\underline{s}^*$ has approximately $(1 - K^{-1})N_3$ zero components. Substituting for K it is seen that $\underline{s}^*$ has approximately

$$\left\{1 - \left(\frac{\lambda_{d\ min}}{\lambda_d}\right)^{1/5}\right\} N_3 \qquad \cdot 3$$

components which are adjacent and approximately zero (to within the accuracy set by θ_2).

·545 Choice of N_3

There are two factors governing the choice of N_3:-

1) In §·5422 and §·543 it was seen that N_3 must be chosen such that

$\lambda_d \geq \lambda_{d\ min}$, a requirement arising from the approximation (1b) of §·541.

2) In order that the circular convolution $\underline{\underline{s}}^* \hat{\underline{p}}$ is a good approximation to the function $s * \hat{p}$, N_3 must be chosen so that it is greater than the total number of non zero terms in $\underline{s}^*$ and $\hat{\underline{p}}$ (isolated zeros being excluded).

From §·5421 λ_c is independent of N_3 and by definition $\lambda_c = \lambda_d \Delta R^n$. Denote by $\lambda_{dN'}$ the value of λ_d when data processing is performed with $N_3 = N'$ points, then

$$\lambda_{dN_3}(\Delta r N_3)^{-5} = \lambda_c = \lambda_{dN_2}(\Delta r N_2)^{-5}$$

hence

$$\lambda_{dN_3} = \left(\frac{N_3}{N_2}\right)^5 \lambda_{dN_2} \qquad \cdot 1$$

and

$$\lambda_{dN_3} \geq \lambda_{d\ min} \qquad \cdot 2$$

is satisfied if $\left(\frac{N_3}{N_2}\right)^5 \lambda_{dN_2} \geq \lambda_{d\ min}$

i.e. if $$N_3 \geq \left(\frac{\lambda_{d\ min}}{\lambda_{dN_2}}\right)^{1/5} N_2 \ . \qquad \cdot 3$$

(Note the implicit assumption in E·2 that $\lambda_{d\ min}$ is independent of N_3: a fact which may be seen from §·5422.) Thus by calculating a trial value of λ_d on the basis of $N_3 = N_2$ data points it is then possible to estimate whether this is sufficient or whether one must use more points and further, on the basis of the original estimate λ_{dN_2} a minimum value of N_3 can be estimated for the new number of points.

From §·544 the number of non zero points in $\underline{s}^*$ is

$$\left(\frac{\lambda_{d\ min}}{\lambda_{dN_3}}\right)^{1/5} N_3$$

and from E·1 this is equal to

$$\left(\frac{\lambda_{d\ min}}{\lambda_{dN_2}}\right)^{1/5} N_2 .$$

From the relations between p,q,g,μ and the original object being scanned, the number of non zero terms in $\hat{\underline{p}}$ should be of order N (see §3·424) or less. Thus the second requirement gives

$$N_3 \geq N + \left(\frac{\lambda_{d\ min}}{\lambda_{dN_2}}\right)^{1/5} N_2. \qquad \cdot 4$$

Since N is positive this condition guarantees that E·3 is true. Once again having calculated a trial value for λ_d with N_2 data points a final value of N_3 may be estimated.

In practice the original estimates $\hat{\underline{p}}(\phi_j)$ must be padded out by adding $(N_3 - N_2)/2$ zero elements at the beginning and end of each $\hat{\underline{p}}(\phi_j)$ (c.f. §·321).

·546 General discussion: n = 1

In the discussion of §§·541 - ·545 various results were obtained for the case $n > 2$, but for the reasons outlined below these cannot be easily extended to cover $n = 1$ (the other case of interest arising in §·521, see also §·55). The form of $S(R)$ for $n = 1$ shows immediately that s does not 'suppress' but only 'fails to enhance' the effects of noise in the data and it is therefore to be expected that the results of data processing with $n = 1$ will probably be less useful. For this reason a set of results parallel to those of §§·541 - ·545 but for the case $n = 1$ has not been developed.

The problems for $n = 1$ arise from the form of $S(R)$, consider first its definition

$$S(R) = \frac{\lambda_c\ |R|}{|R| + \lambda_c} \ .$$

clearly $S \to \lambda_c$ as $|R| \to \infty$ and does not have a Fourier transform in the classical sense. One can attempt to get round this problem by splitting S into two parts giving

$$\lambda_c \mathcal{F}_1(\delta(r)) - S(R) = \frac{\lambda_c^2}{|R| + \lambda_c}$$

(where $\delta(r)$ is the delta function) and then treating the 'classical' part and the 'distribution' part separately; but this still gives problems. The function

$$\frac{\lambda_c^2}{|R| + \lambda_c}$$

is in $\mathcal{L}^2$ and therefore has a F.T. but it is not in $\mathcal{L}^1$, so that attempting to evaluate the transform by straightforward numerical integration (using the D.F.T.) is a waste of time. In particular if the resolution

$$\int_0^\infty \frac{\lambda_c^2}{|R| + \lambda_c}\,dR = \int_0^X \frac{\lambda_c^2}{|R| + \lambda_c}\,dR + \int_X^\infty \frac{\lambda_c^2}{|R| + \lambda_c}\,dR,$$

which is fundamental to the development of §·542 and §·543, is attempted disaster strikes as the last integral is divergent for all X so that there is no such thing as the 'negligible area under the tail of the function'. With the demise of §·542 and §·543 goes also §·544 and §·545 which depend on them and it is clear therefore that the whole of §·54 must be reshaped if one wishes to deal with the case $n = 1$. For this and other reasons, it is taken no further.

·55 Discussion

Four possible formulations have been given for the problem of inverting the Radon transform. It has been shown (§1·1·462) that if

$$p = \mathcal{R}f$$

then f may be expressed as a function of p by

$$f = \mathcal{B}\,\mathcal{F}_1^{-1}|R|\mathcal{F}_1 p\ .$$

As a result of reconsidering the inversion problem in terms of constrained optimization it is seen that the function $|R|$ in the Back projection theorem must be modified to

$$\frac{\lambda_c |R|}{|R|^n + \lambda_c}$$

where n has the values 1,5,0 or 4 for Cases 1 - 4 respectively. The effects of applying the D.F.T. to this function have been considered, and these result in a range of acceptable values for λ_c, and also in an expression for N_3. However, the implications of the acceptable range for λ_c have more consequences which will be discussed in §3.

Of the various values of n only one is taken further: n = 0 (§·523) produces a trivial result, n = 4 (or 0) gives rise to difficulties in assigning a value to e_d^2 (§·5332), and n = 1 (although not ruled out) gives rise to numerical problems (§·546). For this reason only n = 5 is used.

At this point it is instructive to consider how the inversion procedure for $\mathcal{R}$ which has been outlined above (§·5) fits in with the discussion of §1. In §1·6221 it was pointed out that the s filter should be chosen in such a way that its cut-off frequency was matched to the data statistics and this is precisely what has now been achieved. As will be seen in §3·3121, λ_c effectively controls the cut-off frequency of the s filter and in §3·32 E·5 it will be shown that the value of λ_c is controlled by the value of ℓ in such a way that as ℓ increases (better data statistics) λ_c (and hence the cut-off frequency of the s filter) increases. Finally consider how this compares with the work of Chessler outlined in §1·41 (Chessler et al, 1975). In addition to suggesting the Hanning window Chessler also suggests that the cut-off frequency of the window should be related to the radial sampling interval in a fixed way (by putting $\omega_c = 1/4\Delta r$). The effect of this is that the picture produced from the data will not in general be the best possible unless the sampling interval and hence the s_3 cut-off frequency just happens to match the data statistics. In contrast by breaking the link between sampling interval and s filter cut-off frequency the above procedure remains free to choose the cut-off best suited to the data. The question of choosing the best Δr is dealt with in §3.

·6 Discussion

The process of calculating linear attenuation coefficients from scan data has been considered in some detail. In particular, the scan data arising from a translate/rotate machine using an isotope source has been examined and this has lead to the formulation of the mathematical model of §·1. This model can be conveniently viewed as being the composite of four distinct mappings and the methods of inverting each of these has been examined in §·2 -§·5. The result of this is that by following each of the invertion procedures in turn, one can proceed from measurements of the scan data to estimates of the attenuation coefficients. The results of this chapter therefore answer the question "How should one process the scan data?", in the next chapter some of the ideas developed in this chapter together with some new ideas will be used to answer the question "What scan data should one collect?".

(It is interesting in passing to note the implications of the model given in §·1 for the discussion on restoration given by Bracewell (Bracewell, 1977). This model is made up of four mappings which are, in order of application, Radon transform, exponentiation, smoothing, and multiplication by a constant and it should be noted that the exponentiation and the smoothing do not commute. In the discussion on restoration Bracewell assumes that the smoothing effect acts on the line integrals or, equivalently, on the original f. Essentially this means that the process of data collection followed by the log mapping can be taken to be linear and thus can be characterised by a single point spread function for which restoration may be performed. However, the fact that the smoothing occurs after the exponentiation in the model of §·1 suggests that the assumption of linearity is not valid. This has significant implications for the idea of restoration, in particular the following. With the assumption of linearity and the application of restoration to the p data (or equivalently f data), the restoration may be conveniently applied in practice as part of the filteration performed prior to back-projection and the computational overheads are relatively minor. In contrast the model given here suggests that the restoration must be applied before the log mapping and this means a significant computational overhead which may well be unacceptable in practice. Clearly these facts need further consideration.)

Chapter 3 : Prediction Problems

·1 Introduction

In this chapter the aim is to give an answer to the question "What is the best scan data to collect?" It has already been seen in chapter 2 just how the data may be processed but it is obvious that if the correct data is not obtained in the first place then the processing may be a complete waste of time.

During the course of discussing the data processing algorithm, four constants were introduced which represent decisions made during the design and building of the scanner; these were

Scanner parameters

ℓ : the number of counts/cell in air

2a : the collimator diameter

Δr : the cell size

M : the number of traverses during scanning

and in order to proceed with scanner construction one must allocate values to these constants. This therefore raises the question of what values one should use and, not surprisingly, this depends on the quality of the section one wishes to obtain. It would appear then, that a precursor to building the scanner is to specify the quality of section to be reproduced and for present purposes this is taken to be specified by three further parameters

Patient parameters

D : the diameter of the section to be scanned

the spatial resolution of the reconstructed section

the density resolution of the reconstructed section.

In addition, before being able to assign values to the scanner parameters it will be found necessary to assign

m : see §2·543

θ_1 : see §3·2

θ_2 : see §2·543

L : the traverse length, see §3·424

In practice one can expect to have some idea of the range of typical sections (f's) which will be encountered, and some idea of the density and spatial resolutions required in the final reconstruction. The object is therefore to predict the scanner parameters in terms of the patient parameters.

Finally one major point is considered. In chapter 2 it was shown that the scanning process consisted of four parts represented by the mappings

$$f \to p,\ p \to q,\ q \to h * q = g \quad \text{and} \quad g \to \mu$$

and the data processing algorithm was derived to invert each of these mappings in turn (in the reverse order). It was seen that inverting $p \to q$ and $g \to \mu$ is simple while the inversion of $f \to p$ and $q \to h * q$ causes problems due to the discontinuity of the inverse operators. However, it is debatable whether it is worth doing the deconvolution at all as the result is simply to improve the quality of the estimate $\underline{\hat{q}}$ and this could be accomplished just as effectively by collecting better data in the first place. The options are therefore

1) to collect lower quality data and then improve it by invoking computing power to invert $q \to h * q$, or
2) to collect higher quality data and leave out the deconvolution, that is set $\underline{\hat{q}} = \underline{\hat{g}}$,

and this is something that the designer of a scanner must decide for himself on the basis of such factors as patient dose, scanner and computer hardware required, scanning and computing times, etc..

In this chapter it is assumed throughout that one is to collect higher quality data and omit the deconvolution - this has been assumed because, in the author's opinion, it is the most likely choice in practice. It should be admitted that there may be personal bias involved as the author has examined the possibility of deriving results analogous to those contained in this chapter but for the case where deconvolution is performed and the results obtained are of sufficient complexity to make them of doubtful use in practice. (They amount really to a complete computer simulation of the model given in §2·1). The complete algorithm (with and without deconvolution) may be found summarised in §5.

·2 Aliasing and the choice of Δr

When collecting scan data the ideal would be to measure q, but as a result of having to use a large enough collimater and long enough cells to obtain statistically significant results, one measures not q but $g = h * q$ (strictly, one measures $\mu = \ell g$ where ℓ is a constant) where

$$h = h_1 * h_2 \qquad \text{or} \qquad H = H_1 \times H_2$$

and

$$H_1 = \frac{J_1(2\pi aR)}{\pi aR}$$

and

$$H_2 = \frac{\sin(\pi\Delta rR)}{\pi\Delta rR}$$

(see §2·3 E·2 and E·9)

As a further complication, $g(r)$ is not measured for all r but only at equally spaced sample points. If $G(R)$ denotes the C.F.T. of g then the C.F.T. of the sampled data

$$\sum_j \quad g\left((j - \tfrac{N-1}{2})\Delta r\right) \delta\left(r - (j - \tfrac{N-1}{2})\Delta r\right)$$

is

$$\Delta r^{-1} \sum_\ell e^{-i\pi\ell(N-1)} G(R + \ell\Delta r^{-1}) \qquad \text{(see Ap 1·1·32 E·1)}$$

and since this is to be used as an estimate of $G(R)$ for $0 \le R \le 1/2\Delta r$ it is necessary to choose a and Δr such that

$$\sum_\ell \quad e^{-\pi i\ell(N-1)} G(R + \ell\Delta r^{-1}) \simeq G(R) \quad 0 \le R \le \frac{1}{2\Delta r}$$

The choice of $0 \le R \le 1/2\Delta r$ arises because a sample spacing of Δr implies that any frequency components beyond $1/2\Delta r$ are inaccessible.

In practice the Fourier transform is implemented using a D.F.T., however since allowance has been made for the sampling of g the D.F.T. is simply the sampled version of

$$\sum_{\ell} e^{-\pi i \ell (N-1)} G(R + \ell \Delta r^{-1})$$

and the following discussion of aliasing applies to either. The discussion is developed in terms of the continuous situation because the discrete version is difficult to write down despite being straightforward to visualise.
Suppose first that $Q(R) = 1$, then it is sufficient to choose a and r such that

$$\sum |H(R + \ell\Delta r^{-1})| \simeq |H(R)| \qquad 0 \le R \le 1/2\Delta r \qquad \cdot 1$$

The functions H_1, H_2 and H are illustrated in F·1 to F·4 for various Δr and a. The function $\Sigma H(R + \ell\Delta r^{-1})$ is illustrated in F·4 and as Δr becomes smaller relative to a it is clear that the peaks become more narrow and better separated, and that

$$\Sigma\, H(R + \ell\Delta r^{-1}) \to H(R) \quad \text{pointwise for } 0 \le R \le 1/2\Delta r.$$

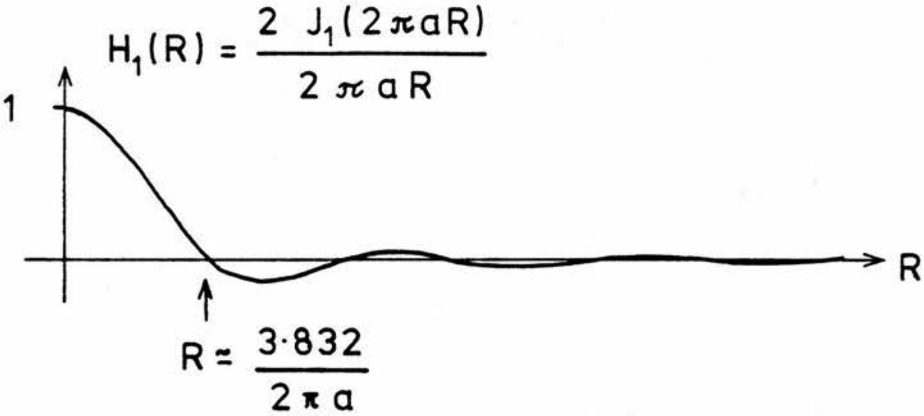
Since $H(R)$ has the form of a damped oscillation about zero, one expects the error introduced by the terms $H(R + \ell\Delta r^{-1})$ in E·1 to become smaller with increasing $|\ell|$. Define the quantity p_ℓ by

$$p_\ell = \max\{|H(R + \ell\Delta r^{-1})| \ : R \in [0, \Delta r^{-1}/2\,]\}$$

$$= \begin{cases} \max\ \{|H(R)| \ : R \in [\ell\Delta r^{-1}, (\ell + \frac{1}{2})\ \Delta r^{-1}]\} & \ell > 0 \\ \max\ \{|H(R)| \ : R \in [(|\ell|-\frac{1}{2})\Delta r^{-1}, |\ell|\Delta r^{-1}]\} & \ell < 0 \end{cases}$$

(The expression for the case $\ell < 0$ is true since H is even.) The dominant contribution to the error arises from the term p_{-1} since in this case H takes values corresponding to $R \in [\Delta r^{-1}/2, \Delta r^{-1}]$. The constant Δr is therefore chosen such that this maximum error is less than a fraction θ_1 of the peak value of H (this peak value is at $R = 0$ where $H(0) = 1$), i.e.

Δr and a are chosen such that $p_{-1} \le \theta_1$.



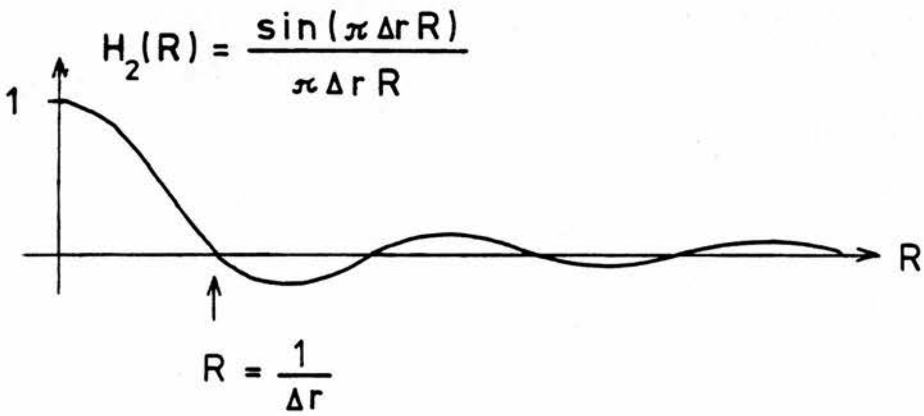
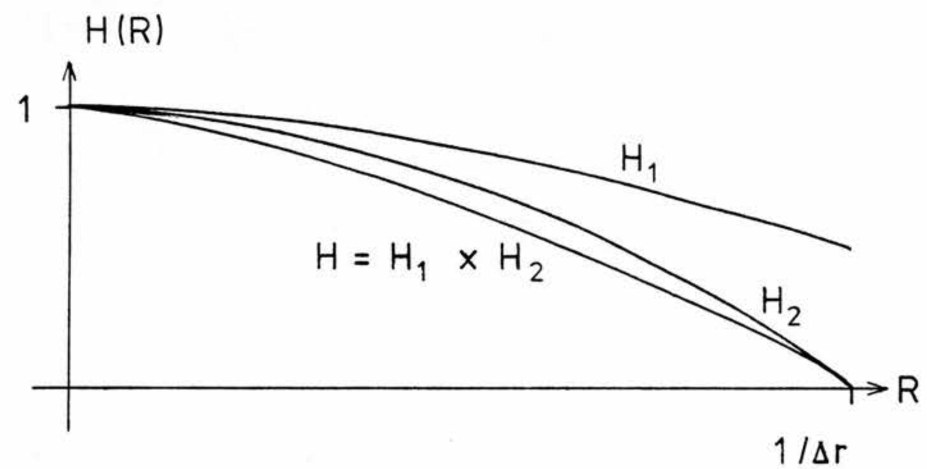


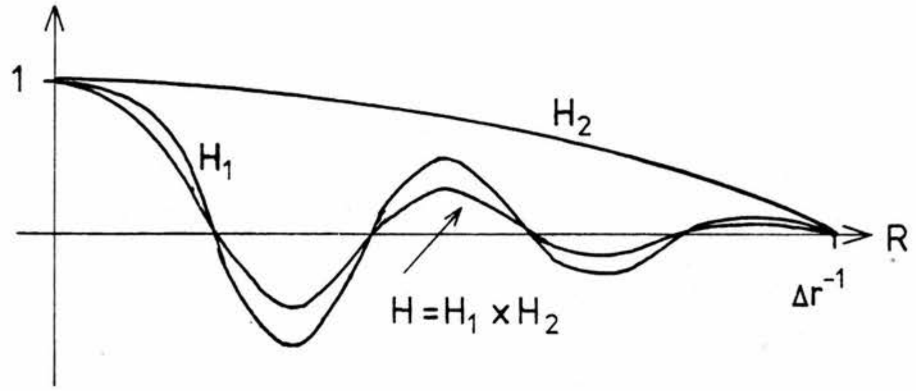
fig. 3·1 : Graph of H_1 and H_2 .

(to scale)



$\Delta r > a$

fig. 3·2 : Graph of H for $\Delta r > a$.
(not to scale)



$\Delta r \simeq a/3$

fig. 3·3 : Graph of H for $\Delta r < a$.

(not to scale)

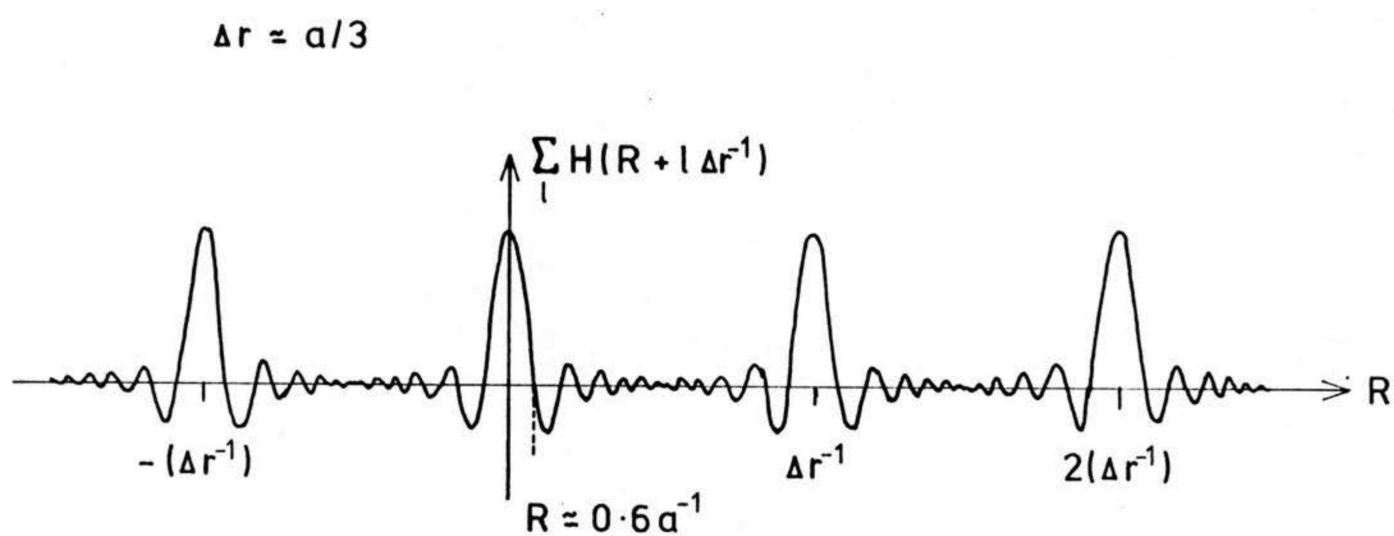


fig. 3·4 : Graph of $\sum_l H(R + l\,\Delta r^{-1})$ for $\Delta r < a$.

(not to scale)

Substituting in p_{-1} for H in terms of H_1 and H_2 and putting $X = 2\pi aR$ and $k = 2a/\Delta r$ this becomes

choose $\Delta r, a$ such that

$$p_{-1} = \max\left\{\left|\frac{2J_1(X)}{X} \; \frac{\sin(k^{-1}X)}{k^{-1}X}\right| \; : X \in [k\pi/2, k\pi]\right\} \le \theta_1 \; .$$

Figure ·5 gives a graph of $p_{-1}(k)$ together with $p_1(k)$ and $p_{-2}(k)$ (the next two terms in order of magnitude) where

$$p_1(k) = \max\left\{\left|\frac{2J_1(X)}{X} \; \frac{\sin(k^{-1}X)}{k^{-1}X}\right| : X \in [k\pi, 3k\pi/2\,]\right\}$$

and

$$p_{-2}(k) = \max\left\{\left|\frac{2J_1(X)}{X} \; \frac{\sin(k^{-1}X)}{k^{-1}X}\right| : X \in [3k\pi/2, 2k\pi]\right\} \; .$$

In addition the curve $k^{-3/2}/2$ is illustrated. Since $p_{-1}(k) \le k^{3/2}/2$ then choosing k such that

$$\frac{1}{2}\,k^{-3/2} \le \theta_1 \qquad \cdot 2$$

ensures that

$$p_{-1}(k) \;\; \le \theta_1 .$$

Equation ·2 is therefore taken as the required constraint on k (and hence Δr and a). Straightforward manipulation of this condition gives

$$(2\theta_1)^{-2/3} \le k$$

or

$$\Delta r (2\theta_1)^{-2/3} \le 2a \qquad \cdot 3$$

In practice $Q(R) \neq 1$, but experience shows that it will usually have the shape of a Gaussian curve. This will improve the situation since any aliased components around $R = (2\Delta r)^{-1}$ will be further reduced.

(The computer program used to calculate p_ℓ together with its output is given in Appendix 3·1).

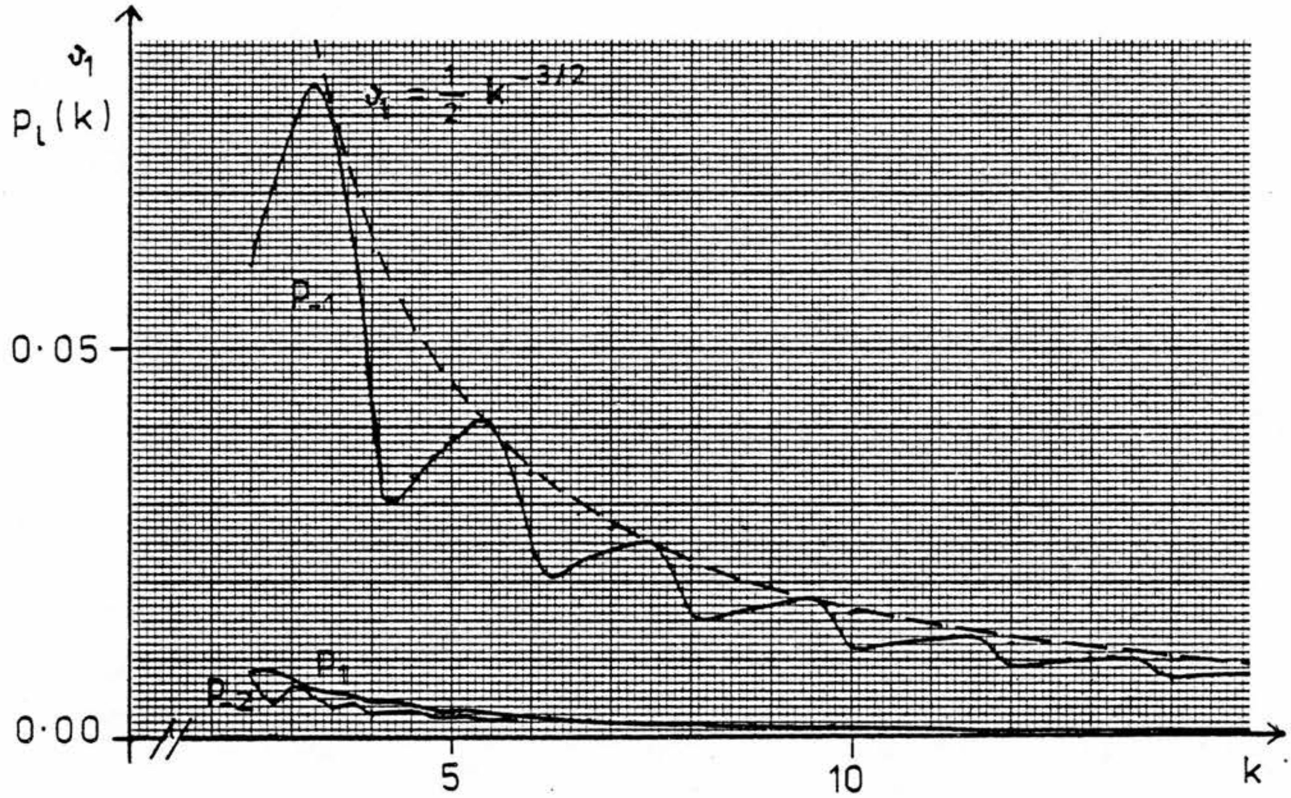


fig. 3·5 : Aliasing error due to $\underline{h}$.

·3 The concept of cut-off frequencies

In this section the first steps are taken towards obtaining a relationship between the scanner parameters $(a, \Delta r, N, \ell, \Delta\phi)$ and the patient parameters (spacial and density resolution, patient dose, etc.). During the course of data collection and processing, the data undergoes various filtering processes and each of these has its own cut-off frequency ω_c in Fourier space, beyond which spatial frequencies are progressively poorly reproduced. Each of these processes is now examined in turn and expressions given for its cut-off frequency. (Where relevant the standard idea of taking ω_c to be the 3dB point is used.) The section finishes (§·33) with a non-quantitative discussion on the effects of allowing any one of these cut-off frequencies to be significantly larger or smaller than the others.

·31 Definitions of cut-off frequencies

·311 The collimater cut-off frequency, $\omega_{c\ col}$

In the ideal scanner one would like to use infinitely small cell size (Δr) and collimater diameter (2a), and in this situation one would be able to measure $\mu = \ell q$. In practice one has to measure $\mu = \ell h * q$ and the data is thus degraded by the filter h.

Assuming no deconvolution is performed (see §·1) then the cut-off frequency associated with h is defined by

$$\omega_{c\ col} = \inf \left\{ |R| \quad : \ (H\overline{H})(R) = \frac{1}{2} \right\}$$

and using the definition of H this becomes

$$\omega_{c\ col} = \inf \left\{ |R| \quad : \ \frac{\sin(\pi\Delta rR)}{(\pi\Delta rR)} \quad \frac{J_1(2\pi aR)}{2\pi aR} = \frac{1}{2\sqrt{2}} \right\}$$

from which it is clear that $\omega_{c\ col}$ is a function of 2a and Δr. However, using the definition, it is straightforward to show that the quantity

$2\pi a\omega_{c\ col}$ is dependent only on $k = 2a/\Delta r$ rather than $2a$ and Δr separately. The values of $2\pi a\omega_{c\ col}$ are tabulated as a function of k in F·6. Examination of this table further shows that $2\pi a\omega_{c\ col}$ is approximately constant for $k = 2(0{\cdot}5)10$, thus taking the value for $k = 4$ as typical gives

$$2\pi a\omega_{c\ col} \simeq 1{\cdot}55930$$

or

$$\omega_{c\ col} \simeq 1/4a.$$

For this reason the value of $\omega_{c\ col}$ is taken for all future purposes to be defined by

$$\omega_{c\ col} = \frac{1}{4a}\ . \qquad \cdot 1$$

Note: 1) this cut-off is described as the collimater cut-off frequency for the reasons outlined in §·24.

2) the details and software used to tabulate $2\pi a\omega_{c\ col}$ are given in Appendix 3·2.

3) if deconvolution is performed then $\omega_{c\ col}$ is defined as in Appendix 3·3.

·312 The s filter cut-off frequency, $\omega_{c\ fil}$

·3121 Continuous case

It was seen in §1·1·462 that

$$f = \mathcal{B}\,\mathcal{F}_1^{-1}|R|\,\mathcal{F}_1 p$$

and in §2·522 and §2·55 that as a result of the discontinuity of $\mathcal{R}^{-1}$ and the presence of noise in $\hat{p}$ that $\hat{f}$ had to be estimated as

$$\hat{f} = \mathcal{B}\,\mathcal{F}_1^{-1}\left(\frac{\lambda_c|R|}{|R|^5 + \lambda_c}\ \mathcal{F}_1\hat{p}\right).$$

TABULATED VALUES OF f(k) = 2*PI*a*OMEGA c col(k)

WHERE :- OMEGA c col IS THE CUT-OFF FREQ. DUE TO THE
COLLIMATOR WHEN NO DECONVOLUTION IS PERFORMED
AND k IS THE NORMALISED DIAMETER 2*a/DELTA r

Normdiam	f(k)
2.000	1.41504
2.500	1.47973
3.000	1.51828
3.500	1.54277
4.000	1.55930
4.500	1.57078
5.000	1.57922
5.500	1.58555
6.000	1.59035
6.500	1.59410
7.000	1.59715
7.500	1.59949
8.000	1.60160
8.500	1.60324
9.000	1.60465
9.500	1.60582
10.000	1.60687

fig. 3·6.

Thus as a result of the presence of noise it is necessary to add the filter

$$\frac{\lambda_c}{|R|^5 + \lambda_c}$$

and $\omega_{c\ fil}$ is therefore defined as

$$\omega_{c\ fil} = \inf \left\{ |R| \ : \ \left(\frac{\lambda_c}{|R|^5 + \lambda_c} \right)^2 = \frac{1}{2} \right\}$$

from which $\omega_{c\ fil} = (\sqrt{2} - 1)^{1/5} \lambda_c^{1/5}$ ·2

·3122 Discrete case

Were it practical one would naturally consider using the filter function $\underline{\underline{\Lambda}}$ in place of $|R|$. However as a result of noise in $\hat{\underline{p}}$ it was shown in §2·531 that the filter

$$\lambda_d \underline{\underline{\Lambda}} \ (\underline{\underline{\Lambda}}^5 + \lambda_d \underline{\underline{I}})^{-1}$$

must be used, so that the additional filtering due to the noise is

$$\lambda_d (\underline{\underline{\Lambda}}^5 + \lambda_d \underline{\underline{I}})^{-1} .$$

The discrete cut-off frequency is therefore taken to be the value of k such that

$$\left(\frac{\lambda_d}{\Lambda_{k'}^5 + \lambda_d} \right) \geq \frac{1}{2} \ \forall\, 0 \leq k' \leq k \quad \text{and} \quad \left(\frac{\lambda_d}{\Lambda_{k+1}^5 + \lambda_d} \right) < \frac{1}{2}$$

i.e. such that

$$k' \leq (\sqrt{2} - 1)^{1/5} \lambda_d^{1/5} \ \forall\, 0 \leq k' \leq k \quad \text{and} \quad k + 1 > (\sqrt{2} - 1)^{1/5} \lambda_d^{1/5} .$$

Note that with this definition

$$k \simeq (\sqrt{2} - 1)^{1/5} \lambda_d^{1/5}$$ ·3

and a discrete frequency k corresponds to a continuous frequency $k\Delta R$ thus the continuous cut-off frequency may be written

$$k\Delta R \simeq (\sqrt{2} - 1)^{1/5} \lambda_d^{1/5} \Delta R$$
$$= (\sqrt{2} - 1)^{1/5} \lambda_c^{1/5}$$

which is in consistent with E·2.

·313 The angular cut-off frequency, $\omega_{c\ ang}$

It is well known that an object with maximum diameter D is completely represented by its sampled Fourier transform provided the sample spacing is less than D^{-1}. (Bracewell, 1965: Ch 10). From the projection theorem (§1·1·444) $(\mathcal{F}_1 p)(R,\phi_j) = F(R,\phi_j)$, i.e. the Fourier transform of $p(r,\phi_j)$ (with respect to r) gives values for $F(R,\phi)$ along the radial lines $\phi = \phi_j$ in Fourier space. At a radius R the spacing between these lines is $R\Delta\phi$. A body of maximum diameter D is therefore considered to be correctly represented only at frequencies R for which this angular spacing is less than or equal to D^{-1}, i.e. for frequencies R such that

$$R\Delta\phi \leq D^{-1} \qquad \cdot 35$$

The cut-off frequency due to angular sampling, $\omega_{c\ ang}$, is therefore defined by

$$\Delta\phi\ \omega_{c\ ang}\ D = 1 \qquad \cdot 4$$

·32 Factors affecting the cut-off frequences

From E·2

$$\omega_{c\ fil} = (\sqrt{2} - 1)^{1/5} \lambda_c^{1/5}$$

where λ_c is determined by (see §2·53, E·5 and E·8)

$$\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dR \left| \frac{|R|^5}{|R|^5 + \lambda_c} \mathcal{F}_1 \hat{p} \right|^2 = e_c^2 = 2\Delta\phi\Delta r \sum_j \operatorname{tr} \underline{\underline{V}}\left(\hat{\underline{p}}(\phi_j)\right).$$

From §5

$$\underline{\underline{V}}(\hat{\underline{p}}(\phi_j)) = \operatorname{diag}(\underline{m})^{-1}$$

hence
$$E\{e_c^2\} = E\left\{2\Delta\phi\Delta r \sum_j \sum_k \frac{1}{m_k(\phi_j)}\right\}$$
$$= 2\Delta\phi \sum_j \Delta r \sum_k E\left\{\frac{1}{m_k(\phi_j)}\right\}$$
$$\simeq 2\Delta\phi \sum_j \Delta r \sum_k \left\{\frac{1}{\ell g_k(\phi_j)}\right\}$$

assuming that the standard deviation of m is small compared to the mean of m, i.e. that $\sqrt{\mu_k(\phi_j)} \ll \mu_k(\phi_j)$ or equivalently $\mu_k(\phi_j) \gg 1$. Hence

$$E\{e_c^2\} \simeq \ell^{-1} \int_0^{2\pi} d\phi \int_{-L/2}^{L/2} dr\ (g(r,\phi))^{-1}$$

where L is the traverse length. By definition $g = h * q$ and g is therefore a smoothed version of q, for this reason one would expect that

$$\int_0^{2\pi} d\phi \int_{-L/2}^{L/2} dr\ (g(r,\phi))^{-1} \simeq \int_0^{2\pi} d\phi \int_{-L/2}^{L/2} dr\ (q(r,\phi))^{-1}$$

and it is seen that λ_c (and hence $\omega_{c\ fil}$) are approximately determined by the equation

$$\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dR \left| \frac{|R|^5}{|R|^5 + \lambda_c} \mathcal{F}_1 p \right|^2 = \ell^{-1} \int_0^{2\pi} d\phi \int_{-L/2}^{L/2} dr\ (q(r,\phi))^{-1}. \qquad \cdot 5$$

(The assumption that p may be substituted for $\hat{p}$ in the left hand side of this equation is justified by the fact that $\sqrt{\mathrm{var}(\hat{p})} \ll E\{\hat{p}\}$. Both this and the earlier requirement that $\mu_k(\phi_j) \gg 1$ are justified at the end of this section.)

Since q and p are both wholly determined by f, E·5 suggests that $\omega_{c\ fil}$ is largely determined by f and ℓ. From E·4 $\omega_{c\ ang}$ is determined by $\Delta\phi$ (or M) and D (where D is determined by f), and from E·1 $\omega_{c\ col}$ is determined by a. Thus by altering ℓ, M and a one can arrive at different values of $\omega_{c\ fil}$, $\omega_{c\ ang}$ and $\omega_{c\ col}$ respectively and in §·33 the effects of doing this are discussed at greater length. However, before doing this, the two statistical assumptions made above are justified.

It will be seen in Chapter 4 that:-

a) the largest object to be scanned is about 30 cm diameter. Assuming that the section is composed of striated muscle then $f \simeq 0{\cdot}085$ and

$$1 \geq q \geq \exp(-0{\cdot}085 \times 30) = 0{\cdot}078.$$

(The value 0·085 for the linear attentuation coefficient is obtained by assuming a density of $1 gm/cm^3$ and a mass attenuation coefficient of 0·085 which may be found by linear interpolation (to 660 kev (Cs^{137})) in the table of values given in (Hubble, 1969) for striated muscle.)

b) ℓ is of the order of 10^4 to 10^6

With these values then $\mu_k(\phi_j)$ will lie in the range $10^3 - 10^6$ giving $\sqrt{\mu} \ll \mu$, which justifies the first assumption. Further (see §5)

$$\dot{E}\{\hat{p}_k(\phi_j)\} \simeq p_k(\phi_j)$$

and $$\dot{E}\{\sqrt{\mathrm{var}(p_k(\phi_j))}\} \simeq \frac{1}{\sqrt{\mu_k(\phi_j)}}$$

and in connection with the second assumption, it is therefore of interest to know when

$$p_k(\phi_j) \gg \sqrt{\frac{1}{\mu_k(\phi_j)}}$$

With the range of values of $\mu_k(\phi_j)$ given above

$$\sqrt{\frac{1}{\mu_k(\phi_j)}} \leq 10^{-3/2}.$$

hence a sufficient condition is $p_k(\phi_j) \gg 10^{-3/2}$. And this will be true (for a section composed of striated muscle) if the section diameter is greater than

$$10 \times 10^{-3/2}/0{\cdot}085 \simeq 4\,(\mathrm{cm}),$$

and this will usually be the case.

·33 Effects of imbalance in the cut-off frequencies

It has already been seen in Chapter 1 that $\mu(r,\phi)$ is sampled not for all r and ϕ but for

$$r = \left(j - \frac{N-1}{2}\right)\Delta r \qquad j = 0(1)N-1$$

and

$$\phi = k\pi/M \qquad k = 0(1)M - 1 \quad .$$

This in turn means that the estimate $\hat{p}$ of p is obtained only for

$$r = \left(j - (N_3 - 1)/2\right)\Delta r \qquad j = 0(1)N_3 - 1$$

and for the same values of ϕ . It follows immediately from the Back projection theorem that if one uses the D.F.T. on $\hat{\underline{p}}\ (\phi_k)$ one can obtain estimates $\hat{F}(R,\phi)$ of F for

$$R = j\Delta R = j/N_3\Delta r \qquad j = -\frac{N_3}{2}\ (1)\ \frac{N_3}{2}$$

$$\phi = k\pi/M \qquad k = 0(1)M - 1$$

where $F(R,\phi)$ is the 2-D Fourier transform of f. The process of scanning and obtaining the estimate $\hat{\underline{p}}(\phi_j)$ is thus equivalent to obtaining estimates $\hat{F}$ of F with the sampling points of F forming a polar grid in Fourier space. The number of points along a radial line being determined by N (and N_2,N_3) and the number of points around a circle being 2M. Given this and the relation between f and F, it will not be surprising if increasing N (N_2,N_3) increases the resolution (or cut-off frequency) of f in the radial direction, while increasing M increases the resolution of f in the azimuthal direction.

Precisely this effect has been illustrated in (Crowther et al, 1972) and (Klug, 1972), though in these papers it is demonstrated in connection with parametric reconstruction methods. Thus in thinking of radial and azimuthal resolution (cut-off frequency) there is a physically identifiable effect being considered not just a theoretical effect which (since it has not been properly defined) may not even exist.

Now consider how these ideas of azimuthal and radial resolution relate to the cut-off frequencies defined in §·31. It is clear that

$\omega_{c\ col}$ and $\omega_{c\ fil}$ are both measures of the radial resolution while $\omega_{c\ ang}$ is a measure of the azimuthal resolution. Each of these is now considered separately.

·331 Radial Resolution

As noted in §·32, $\omega_{c\ col}$ is determined by a while $\omega_{c\ fil}$ is determined mainly by ℓ (and f). It is thus possible to choose a and ℓ so that $\omega_{c\ col} >> \omega_{c\ fil}$, $\omega_{c\ col} \simeq \omega_{c\ fil}$, or $\omega_{c\ col} << \omega_{c\ fil}$.

Suppose $\omega_{c\ col}$ $\omega_{c\ fil}$, a condition which can be assured by putting a and ℓ sufficiently small. Under these conditions, because a has been chosen small, the m-data (and hence the $\hat{g},\hat{q}$ and $\hat{p}$ estimates) will contain high frequency information, but all to no avail as the low value of $\omega_{c\ fil}$ will ensure that the filtering inherent in calculating $s * \hat{p}$ will tend to filter out this information. In this situation it is clear that the radial resolution will be determined principally by $\omega_{c\ fil}$, however if one were to choose a larger collimater (increase a) thus giving more counts/cell (increasing ℓ) then $\omega_{c\ col}$ would decrease and $\omega_{c\ fil}$ would increase and the radial resolution would therefore increase until one approached the point where $\omega_{c\ col}$ and $\omega_{c\ fil}$ were comparable. In physical terms the situation is: by taking a to be small one is choosing to take many closely spaced readings, but because ℓ is also small these readings are of low statistical significance and the high frequency components so carefully measured in the m data will just be measurements of statistical noise which the s filter will (quite rightly) remove. The remedy of increasing a and ℓ corresponds to taking fewer readings of greater statistical significance (i.e. effectively combining adjacent readings from the original a and ℓ), but one must be careful not to go to the opposite extreme.

Suppose $\omega_{c\ col} << \omega_{c\ fil}$, which can be assured by taking a and ℓ

sufficiently large. In this case the radial resolution is going to be determined mainly by $\omega_{c\ col}$ and will be improved if a and ℓ are reduced thereby increasing $\omega_{c\ col}$. It is to be expected that this process could be usefully continued until one approached the case where $\omega_{c\ col}$ is comparable to $\omega_{c\ fil}$ (beyond this point the situation outlined in the previous paragraph will be encountered). The physical interpretation of $\omega_{c\ col} \ll \omega_{c\ fil}$ is as expected: large a and ℓ means that a few readings with good data statistics ensure that $\omega_{c\ fil}$ is high and that s is a high bandwidth filter but this promise of high resolution is not fulfilled as the high frequency components in $\hat{p}$ have already been averaged out of the m data (from which $\hat{p}$ is calculated) by the use of the large collimater (large a).

The inevitable conclusion is that one should try to avoid both of the above pitfalls by choosing a and ℓ so that $\omega_{c\ col}$ and $\omega_{c\ fil}$ are comparable.

·332 Azimuthal resolution

Suppose that $\omega_{c\ col}$ and $\omega_{c\ fil}$ are chosen to be comparable and use either as a measure of the radial resolution. As has been seen in the references quoted above, the effect of choosing the azimuthal resolution (of which $\omega_{c\ ang}$ is a possible measure) significantly different from the radial resolution (of which $\omega_{c\ col}$ and $\omega_{c\ fil}$ are possible measures) is to introduce a point spread function which is non-uniform over the reconstruction area. This suggests (under the assumption that such a point spread function is better avoided) that $\omega_{c\ ang}$ should be chosen comparable to $\omega_{c\ col}$ and $\omega_{c\ fil}$.

·4 Prediction of scanner parameters in terms of $\omega_{c\ rec}$

·41 Definition of $\omega_{c\ rec}$

In §·31 the various cut-off frequencies were defined and the discussion of §·33 has illustrated the effects of failing to match up the three cut-offs correctly. In order to avoid these problems (by ensuring compatability between the different cut-offs) it is now proposed that scanning should be performed in such a way that $\omega_{c\ ang}$, $\omega_{c\ col}$ and $\omega_{c\ fil}$ are approximately equal. The single value for the cut-off frequency which results is denoted by $\omega_{c\ rec}$ and is taken as a measure of the cut-off for the complete scanning and reconstruction process, i.e.

$$\omega_{c\ rec} = \omega_{c\ ang} = \omega_{c\ col} = \omega_{c\ fil} \qquad \cdot 11$$

The reader should be careful not to confuse the naturalness and simplicity of this criterion with its implications which are far reaching. In chapter 2 and § 3·2 many expressions have been obtained for various constants in the reconstruction process, the addition of this single criterion (E·11) is a last link in the reasoning which allows one to mould these results into a complete procedure for the design of a scanner. It should also be borne in mind that E·11 has the status of a postulate (albeit one which is well motivated by the discussion of §·33) not a proved result.

·42 Prediction of scanner parameters

·421 Prediction of M(or $\Delta\phi$)

By definition $\Delta\phi\omega_{c\ ang} D = 1$ (see §·313), hence if $\omega_{c\ ang} = \omega_{c\ rec}$ then

$$M = \pi\omega_{c\ rec} D$$

since $\Delta\phi = \pi/M$. In general this expression will not be integer and M is therefore taken as

$$M = [\pi D\omega_{c\ rec} + 1] \qquad \cdot 12$$

(see Appendix 2·5 for definition of $[x]$).

·422 Prediction of 2a

By definition $4a\omega_{c\ col} = 1$ (see §·311) hence if $\omega_{c\ col} = \omega_{c\ rec}$ then

$$2a = \frac{1}{2\omega_{c\ rec}} \qquad \cdot 13$$

·423 Prediction of Δr

Three factors affect the choice of Δr:-

1) From §·2, if aliasing errors in the g data (i.e. m data) are to be adequately controlled then

$$\Delta r \le (2\theta_1)^{2/3} 2a$$

and if 2a is assigned as in E·13 then

$$\Delta r \le \frac{(2\theta_1)^{2/3}}{2\omega_{c\ rec}} \qquad \cdot 14$$

2) From §2·5431 Δr must be such that

$$\lambda_d \le \lambda_{d\ max} = 2\cdot 1877 \times 10^{-3} \left(\frac{N_3}{m - 1}\right)^5$$

Since $\lambda_d = (\Delta r N_3)^5 \lambda_c$ this becomes

$$\Delta r \lambda_c^{1/5} \le \frac{(2\cdot 1877 \times 10^{-3})^{1/5}}{m - 1}$$

and if $\omega_{c\ fil} = \omega_{c\ rec}$ then substitution of §·3121 E·2 gives

$$\Delta r \leq \frac{(2{\cdot}1877 \times 10^{-3}(\sqrt{2}-1))^{1/5}}{\omega_{c\ rec}(m-1)} \qquad \cdot 15$$

3) In §·32 E·5 it was seen that λ_c (and hence $\omega_{c\ fil}$) was largely determined by ℓ (and f). If one considers scanning for some fixed dose (total number of counts) per traverse then decreasing Δr implies decreasing ℓ and hence $\omega_{c\ fil}$. There is thus a definite disadvantage in choosing Δr smaller than is strictly necessary (apart from the implied increase in computational overheads).

Combining these three factors given above, Δr is therefore assigned as

$$\Delta r = \min\left\{\frac{(2{\cdot}1877 \times 10^{-3}(\sqrt{2}-1))^{1/5}}{\omega_{c\ rec}(m-1)}, \frac{(2\theta_1)^{2/3}}{2\omega_{c\ rec}}\right\} \qquad \cdot 16$$

·424 Prediction of ℓ

By definition (see §2·532), λ_d is determined by

$$\sum_{j=1}^{M}\sum_{k=0}^{N_3-1}\left|\frac{\Lambda_k^5}{\Lambda_k^5+\lambda_d}(\underline{\underline{W}}\,\underline{p}(\phi_j))_k\right|^2 = e_d^2$$

and from the intermediate working of §·32 $\dot{E}\{e_d^2\}$ is given by

$$\dot{E}\{e_d^2\} \simeq \ell^{-1}\sum_{j=1}^{M}\sum_{k=\frac{N_3-N}{2}}^{\frac{N_3+N}{2}-1}(\underline{q}(\phi_j))_k^{-1}$$

(N.B. the limits for k arise as the summation must take place only over the N originally observed data points and not over the additional ones arising from padding out from N to N_2 (see §2·321) or from N_2 to N_3 (see §2·545)).

Thus if the data is to be collected in such a way that the cut-off frequency $\omega_{c\ fil}$ defined by

$$\omega_{c\ fil} = (\sqrt{2}-1)^{1/5}\,\lambda_c^{1/5}$$

is equal to $\omega_{c\ rec}$, then

$$\ell \simeq \frac{\sum_{jk} (q_k(\phi_j))^{-1}}{\sum_{jk} \left| \frac{\Lambda_k^5}{\Lambda_k^5 + \lambda_d} (\underline{\underline{W}}\ \underline{p}(\phi_j))_k \right|^2} \qquad \cdot 17$$

where

$$\lambda_d = \frac{(\Delta r N_3 \omega_{c\ rec})^5}{\sqrt{2} - 1}$$

and the summations are taken over the appropriate limits.

(The value of N causes no problems. Denote by L the length of a traverse which, in any practical situation, will have to be chosen somewhat larger than D. It is assumed, both here and throughout the chapter, that L is assigned. Given L and Δr (as in §·423) then N may be set by

$$N = 2 \times [L/2\Delta r + 1]$$

·43 Computational aspects of prediction procedure

·431 Development of ideas

In §·42 explicit formulae have been given for assigning the parameters M, 2a, Δr and ℓ. The computation of the first three is trivial, but the evaluation of ℓ is more involved and this is now examined in more detail. This work falls into three parts:-

1) the specification of the phantom f
2) choosing suitable $\Delta r'$, N_3 and ΔR for use in calculating (3)
3) giving expressions for $\underline{p}(\phi_j)$ and $\underline{\underline{W}}\ \underline{p}(\phi_j)$.

·4311 Specification of phantom

While the function f may be discrete or continuous, the use of the results of §·42 implies the use of a computer and this in turn requires f to be digitised. For this reason, the starting point is taken as a phantom f(x,y) with the following properties:-

$f(x,y)$ is constant in $(j\Delta r_2, (j+1)\Delta r_2) \times (k\Delta r_2, (k+1)\Delta r_2)$

for $j,k \in \mathbb{N}$.

$f(x,y) = 0$ if $|x| > N_4\Delta r_2/2$ or $|y| > N_4\Delta r_2/2$

where $\Delta r_2 > 0$ and N_4 is a positive integral power of 2.

Such a function is completely described by the N_4^2 values it assumes on the squares given by

$$\left(\left\{j - \frac{N_4+2}{2}\right\}\Delta r_2, \left\{j - \frac{N_4}{2}\right\}\Delta r_2\right] \times \left(\left\{k - \frac{N_4+2}{2}\right\}\Delta r_2, \left\{k - \frac{N_4}{2}\right\}\Delta r_2\right]$$

where $1 \le j,k \le N_4$

(see F·7). The centre point of each square has coordinates

$$\left(\left(j - \frac{N_4+1}{2}\right)\Delta r_2 , \left(k - \frac{N_4+1}{2}\right)\Delta r_2\right) \qquad \cdot 21$$

and the phantom f is therefore specified by the cell size Δr_2 and the $N_4 \times N_4$ matrix f_{jk} defined by

$$f_{jk} = f\left(\left(j - \frac{N_4+1}{2}\right)\Delta r_2, \left(k - \frac{N_4+1}{2}\right)\Delta r_2\right) \qquad 1 \le j,k \le N_4 \qquad \cdot 22$$

(see F·7).

·4312 Choosing $\Delta r'$ and N_3

Given a phantom f with the properties outlined in §·4311 then the associated functions p and F are known. Given a value for λ_c as well then the functions $s * p$ and SF are also known.

However in order to apply the result of §·424 it is necessary to sample these functions giving vectors $\underline{p}(\phi_j)$, $\underline{\underline{W}}\,\underline{p}(\phi_j)$ etc., and the sampling interval $(\Delta r')$ and number of samples (N_3) must be chosen with due regard to the functions or the sampling may not be representative. The constraints on $\Delta r'$ and N_3 are essentially those which have been given in §2·54 in connection with the reconstruction algorithm.

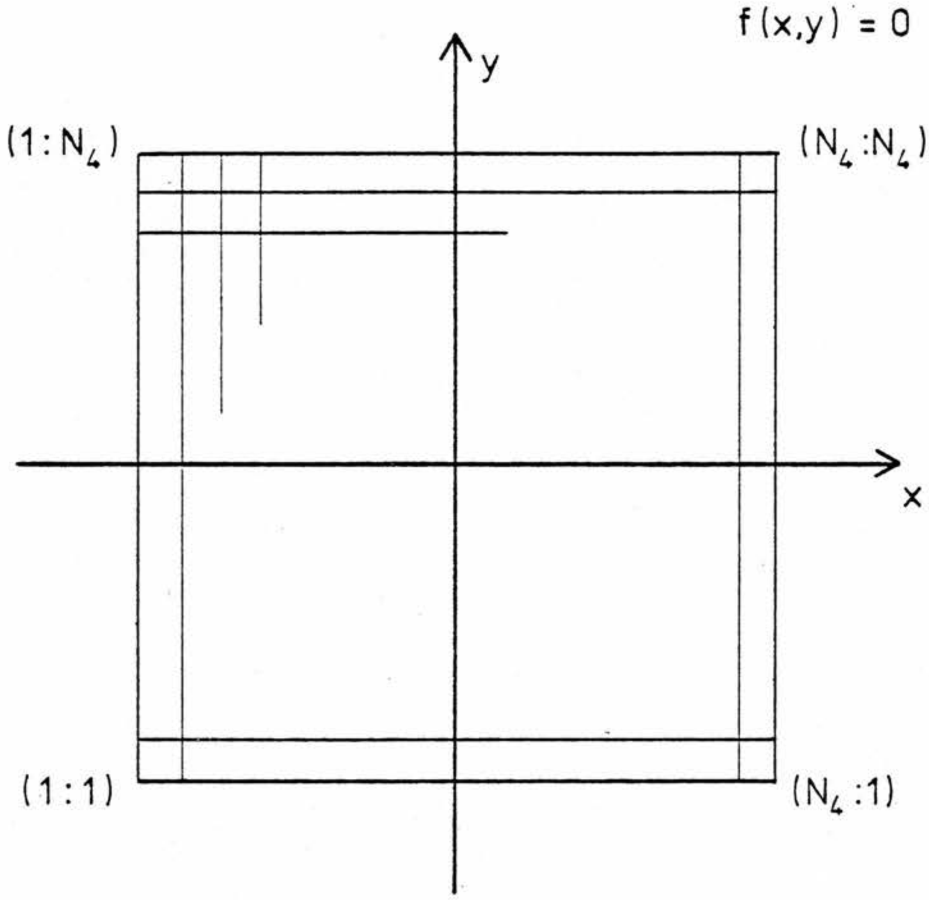


fig. 3·7 : Relation between f_{jk} and f(x,y).

Thus:-

1) In order that

$$\left| \frac{|R|^5}{|R|^5 + \lambda_c} \quad F(R,\phi_j) \right|^2$$

is sampled correctly before performing the summation of §·424 E·17 one has

a) $\Delta r'$ is restricted by

$$\lambda_c \le \lambda_{c\ max} = \frac{2{\cdot}1877 \times 10^{-3}}{\Delta r'^5 (m - 1)^5}$$

which (as in §·423) leads to

$$\Delta r' \le \frac{(2{\cdot}1877 \times 10^{-3} (\sqrt{2} - 1))^{1/5}}{\omega_{c\ rec}(m - 1)} \qquad {\cdot}23$$

b) N_3 is restricted by

$$\lambda_d \ge \lambda_{d\ min}(\theta_2)$$

(see § 2·5431) and substituting in terms of λ_c this leads to

$$\Delta r' N_3 \ge \left(\frac{\lambda_{d\ min}(\theta_2)}{\lambda_c} \right)^{1/5} = \frac{((\sqrt{2} - 1)\lambda_{d\ min}(\theta_2))^{1/5}}{\omega_{c\ rec}} \qquad .24$$

2) In order that

$$\sum (q(r_k,\phi_j))^{-1}$$

should be correctly evaluated:-

a) $\Delta r'$ is restricted by

$$\Delta r' \le \Delta r_2/4 \quad \text{where} \quad \Delta r_2 = \text{phantom cell size} \qquad {\cdot}25$$

(this is so that $(g(r_k))^{-1}$ should represent the overall shape of $(g(r))^{-1}$, the particular factor of 4 being only the author's personal preference)

b) N_3 is restricted by

$$N_3 \ge N$$

so that the summation $\Sigma\, q^{-1}$ can take place over the range of r for which scanning is actually performed (c.f.§·424). Thus

$$\Delta r' N_3 \geq \Delta r' N = L \qquad \cdot 26$$

Combining the results of E·23 to E·26, $\Delta r'$, N_3 and Δr are therefore chosen as follows:-

$$\Delta r' = \min \left\{ \frac{(2\cdot 1877 \times 10^{-3}(\sqrt{2}-1))^{1/5}}{\omega_{c\ rec}(m-1)}, \frac{\Delta r_2}{4} \right\} \qquad \cdot 27$$

$$N_3' = \frac{1}{\Delta r'} \max \left\{ \frac{((\sqrt{2}-1)\lambda_{d\ min}(\theta_2))^{1/5}}{\omega_{c\ rec}}, L \right\}$$

$$N_3 = \inf \{2^n : n \in \mathbb{N} \text{ and } 2^n \geq N_3'\} \qquad \cdot 28$$

$$\Delta R = (\Delta r' N_3)^{-1}$$

·4313 Calculation of $\underline{p}$ and $\underline{\underline{W}}\,\underline{p}$

Given a phantom f_{jk} as specified in §·4311 together with $\Delta r'$ and N_3 as in §·4312 then $\underline{p}(\phi)$ may be evaluated as

$$(\underline{p}(\phi))_\ell = \sum_{(j,k)\in A(\ell)} f_{jk} p^*(\text{dev}(j,k,\ell,\phi))$$

(see Appendix 3·4 for details). The transform $\underline{\underline{W}}\,\underline{p}$ may be calculated using an F.F.T. algorithm in the usual way.

·432 Complete computational scheme for prediction

In this section, the complete prediction procedure, which has been the main aim of this chapter, is summarised. In order to get started the user must specify various constants (§·4321) including $\omega_{c\ rec}$, the reconstruction resolution, which is essentially a combination of the density and spacial resolutions exhibited in §·1 although this relationship has not yet been fully developed (see §·5). Once these constants are assigned then the scanner parameters may be calculated as in §·4322. A complete software listing may be found in Appendix 3·5.

·4321 Information required prior to prediction

·43211 f_{jk}, Δr_2,N_4 - phantom or section.

A complete array f_{jk} (together with Δr_2, N_4) representing the phantom or section to be scanned must be given, this is assumed to be of the form specified in §·4311.

·43212 L - length of scan

In a practical scanner the traverse length will be somewhat longer than the diameter of the phantom, and this length must be specified.

·43213 $\omega_{c\ rec}$ - reconstruction cut-off frequency.

As discussed in §§·3 and ·41 this parameter is essentially a measure of the overall resolution of the scanning and reconstruction process, its relation to the patient parameters is discussed in §·52.

·43214 m - minimum number of samples in first peak of s.

See §2·543 for definition. Bearing in mind the roughly sinusoidal shape of s values of m of about 5 - 10 would seem reasonable.

·43215 θ_1 - maximum aliasing error in m-data.

See §·2. This constant must be chosen so that the errors due to aliasing relate sensibly to the data statistics and to density resolution required in the final reconstruction. See also §·43216.

·43216 θ_2 - maximum aliasing error in filter s.

See §2·543. As for θ_1, this must be chosen so that errors due to aliasing relate sensibly to the density resolution required. However there is an important distinction between θ_1 and θ_2 that must be borne in mind: θ_1 essentially determines the relation between 2a and Δr and thus affects the whole machine design, engineering tolerances etc. while θ_2 only determines N_3 during the data processing. For this reason there are heavy penalties involved in choosing θ_1 unnecessarily small while θ_2 may be chosen too small ("just to on the safe side") with relative impurity. (In the prediction software to be given, $\theta_2 \simeq 1\%$ at all times, giving $\lambda_{d\ min}(\theta_2) \simeq 2000$.).

·4322 Summary of steps in prediction procedure.

·43221 D - diameter of phantom

This must be calculated before M may be assigned. It is not really a part of the prediction itself but rather a detail which must be calculated before prediction can proceed.

From §·4311, the continuous $(f(x,y))$ and discrete (f_{jk}) versions of the phantom are related by E·22 so that if (x,y) are the coordinates of the centre of a phantom element, then

$$x = \left(j - \frac{N_4 + 1}{2}\right)\Delta r_2$$

$$y = \left(k - \frac{N_4 + 1}{2}\right)\Delta r_2$$

and with these values of x,y it is natural to define D as

$$\sup\{\sqrt{x^2 + y^2} : f(x,y) > 0\}.$$

The definition

$$D = \sup\left\{\Delta r_2\sqrt{\left(j - \frac{N_4 + 1}{2}\right)^2 + \left(k - \frac{N_4 + 1}{2}\right)^2} : 1 \le j,k \le N_4 \text{ and } f_{jk} > 0\right\}$$

is therefore taken as the discrete definition used in practice.

·43222 2a - collimater diameter

This may now be predicted as

$$2a = \frac{1}{2\delta_{c\ rec}}$$

(see §·422).

·43223 Δr - cell size

This may now be predicted as

$$\Delta r = \min\left\{\frac{(2\cdot 1877\times 10^{-3}(\sqrt{2}-1))^{1/5}}{\omega_{c\ rec}(m-1)}\ ,\ (2\theta_1)^{2/3}2a\right\}$$

(see §·423)

·43224 M - the number of traverses

This may be predicted as

$$M = [\pi D\omega_{c\ rec} + 1]$$

·43225 ℓ - the number of counts/cell

This may be predicted as follows:-

set $$\Delta r' = \sin\left\{\frac{(2\cdot 1877\times 10^{-3}\ (\sqrt{2}-1))^{1/5}}{\omega_{c\ rec}(m-1)}\ ,\ \frac{\Delta r_2}{4}\right\}$$

and $$N'_3 = \frac{1}{\Delta r'}\ \max\left\{\frac{((\sqrt{2}-1)\lambda_{d\ \min}(\theta_2))^{1/5}}{\omega_{c\ rec}}\ ,\ L\right\}$$

and $$N_3 = \inf\{2^n : n\in\mathbb{N}\ \text{and}\ 2^n \geq N'_3\}$$

(see §·4312).

set $$(\underline{p}(\phi))_\ell = \sum_{(j,k)\in A(\ell)} f_{jk}\ p^*(\mathrm{dev}(j,k,\ell,\phi))$$

and $$\underline{F}(\phi) = \underline{\underline{W}}\ \underline{p}(\phi)$$

where $\phi = \phi_j = j\frac{\pi}{M}$ for $j = 0(1)M-1$ (see §·4313). Then set $$S_1 = \sum_{j=1}^{M} \sum_{k=\frac{N_3-N}{1}}^{\frac{N_3+N}{2}-1} e^{p_k(\phi_j)}$$

and $$S_2 = \sum_{j=1}^{M} \sum_{k=0}^{N_3-1} \left| \frac{\Lambda_k^5}{\Lambda_k^5 + \lambda_d} (\underline{\underline{W}}\ \underline{p}(\phi_j))_k \right|^2$$

where $N = 2 \times [L/2\Delta r + 1]$ (see §·424).

With these preliminaries ℓ may be predicted from

$$\ell = S_1/S_2 \ .$$

·5 Error estimates for $\hat{f}$

In this section the mathematics of the thesis is concluded by a brief discussion of the error in $\hat{f}$. The correct development of the various expression in §·51 requires a mixture of continuous and discrete reasoning in much the same way as §2·5, however rather than confuse the issue by putting in a number of lengthy discrete formulae which are not very easy to assimilate the discussion is presented in continuous terms only. The reader who has grasped the details of the change over from the continuous development to the discrete conclusion in §2·5 will have no difficulty in filling in the correct details.

·51 Development and interpretation of error expressions

It has already been seen in §§2·522, 2·541 and 2·55 that f is estimated by

$$\hat{f} = \mathscr{B}(s * \hat{p}).$$

In order to consider the difference $\hat{f} - f$, $\hat{p}$ is split up into three parts

$$\hat{p} = p + E\{\hat{p} - p\} + \varepsilon$$

where this equation is taken as the definition of ε. It follows from the linearity of $\mathscr{B}$ and the properties of convolution that

$$\hat{f} - f = \{\mathscr{B}(s * p) - f\} + \{\mathscr{B}(s * E\{\hat{p} - p\})\} + \{\mathscr{B}(s * \varepsilon)\}.$$

Each of these three terms is now discussed in detail.

·511 Regularisation error

In §1·1·462 the relation

$$f = \mathscr{B}\mathscr{F}_1^{-1}|R|\mathscr{F}_1 p$$

was derived but because of the discontinuity of $\mathscr{R}^{-1}$ (see §1·1·482) this is of little use in numerical work. Accordingly the relation

$$\hat{f} = \mathcal{B}\,\mathcal{F}_1^{-1} \frac{|R|\lambda_c}{|R|^5 + \lambda_c}\,\mathcal{F}_1 p$$

was derived in §2·5 as a numerically stable approximation to $\mathcal{R}^{-1}$. This process of changing from an exact but unstable operator to an approximate stable operator introduces its own error which is expressed in the first error term. (Such a substitution is sometimes called "regularisation", see (Tikhonov et al, 1977)).

From §3·6·1

$$\mathcal{B}\,(s * p) - f = \mathcal{F}_2^{-1}\left(\frac{-|R|^5}{|R|^5 + \lambda_c}\right) * f$$

which represents a high pass filtering applied to f, so that the error consists of removing the fine detail from the reconstruction.

Note that since this term is dependent only on λ_c (and f) it represents nothing but the error introduced by regularising $\mathcal{R}^{-1}$.

·512 Collimater and regularisation error

In §3·6·2 it is seen that the second error term

$$\mathcal{B}\,(s * E\{\hat{p} - p\})$$

can be identified with

$$\mathcal{B}\,(s * \ln(q/g)).$$

This term depends on the relation between g and q as well as on λ_c. If an ideal collimater $(h(r) = \delta(r))$ were possible then g would be equal to q and $\ln(q/g)$ equal to 0 giving a zero error term. Thus this term represents the effect of the smoothing introduced by the collimater followed by the regularisation of $\mathcal{R}^{-1}$.

·513 Statistical errors

In §3·6·3 it is seen that the third error term has the properties:-

1) $E\,\{\mathscr{B}(s * \varepsilon)\}$ is zero

2) $\mathrm{var}\,\{\mathscr{B}(s * \varepsilon)\}$ may be identified with $\ell^{-1}\mathscr{B}(s^2 * (g)^{-1})$.

Thus while the first two error terms represent systematic errors, this term represents the random component of the error arising from the original data statistics (ℓ) and further affected by the collimater (g) and the regularisation (s).

·52 Relation between errors and patient parameters

It is clear that when one looks at a reconstruction the factor which determines the smallest discernable density deviation over an area is the amount of random noise present in the area, thus the third error term in §·51 basically determines the density resolution of the section.

Similarily the systematic error represented by the first two terms will determine the spatial resolution in the section.

Now consider what determines the size of these error terms. If the prediction procedure of §·4 has been followed then:-

1) The first term is determined by λ_c (and f) and hence by $\omega_{c\ rec}$.
2) The second term is determined by λ_c, 2a and to a lesser extent Δr (and f) and each of these is determined by $\omega_{c\ rec}$.
3) The third term is determined by λ_c, 2a, and to a lesser extent Δr (and f) and again each of these is set by $\omega_{c\ rec}$.

Thus all three error terms will depend only on $\omega_{c\ rec}$ (and f) and there is therefore no reason why these errors should not be calculated as part of the prediction procedure. In such a case one could choose f and $\omega_{c\ rec}$, calculate all the scanner parameters and calculate the error functions and examine them visually, and, on the basis of this visual examination, decide whether the density and spatial resolution obtained were sufficient or whether it is necessary to choose a higher value of $\omega_{c\ rec}$.

(Such an incorporation of the errors into the prediction process has not been attempted by the author due to lack of time, however it is clear that most of the details required have already been developed in §2·531 and §·4).

In connection with statistical errors the following papers are relevant (Barret et al : 1976) (Gore et al : 1978) (Huesman : 1977) (Riederer : 1978) (Tanaka : 1975a) (Tanaka et al : 1975b) (Tanaka et al : 1976). The author is unaware of any papers on the systematic errors of §·511 and §·512.

·6 Discussion

The aim of this chapter has been to discuss how the scanner parameters may be assigned values when only the patient parameters are known (see §·1).

As a preliminary a relation between Δr and $2a$ is derived (§·2) before introducing the idea of cut-off frequencies in §·3. The discussion of cut-off frequencies leads naturally to the proposal that all cut-off frequencies should be set equal to $\omega_{c\ rec}$ (see §·41) and from this and various relations previously derived it is then possible to give formulae for all the scanner parameters in terms of $\omega_{c\ rec}$ (§·4). Finally, there remains the question of relating the spatial and density resolution to the scanner parameters. By considering errors (§·5) it is shown (in principle) how the density and spatial resolution may be evaluated in terms of $\omega_{c\ rec}$. This then completes the chapter, for one can then choose $\omega_{c\ rec}$ and evaluate the resulting density and spatial resolutions, if these are inadequate a better value of $\omega_{c\ rec}$ can be chosen, this process can be repeated until the correct value of $\omega_{c\ rec}$ is established and the scanner parameters may then be assigned as detailed in §·4.

The procedure outlined above has one major deficiency and that is that one must specify a phantom before the prediction of the scanner parameters proceeds. Thus one might say that it is necessary to know the answer before one can start. This however is not the whole truth. One could ask "How do you solve an equation?" to which the immediate reply is "Which equation?". Similarily if one asks "How do you build a scanner?" then this raises the question "What sections do you want to look at?" - looked at in this light it is hardly surprising that the prediction procedure makes reference to f. What is disappointing is that the procedure requires the complete specification of f and that practical use of the procedure implies that one must use it with a range of f's intended to represent all sections likely to be encountered on the final machine. What would be attractive would be to show that the various expressions dependent on f in the prediction procedure were (for practical purposes anyway) dependent only on one or two gross characteristics of f (e.g. mean density, diameter, etc.). The author has made some brief attempts to identify such characteristics and these were not successful, but it should be understood that these attempts were by no means exhaustive.

Chapter 4: Equipment

·1 Introduction

It has already been seen in the Preface, §1 and §2 that the basic requirements are:-

a) an γ-ray source and detector.

b) the necessary machanics to position the γ-ray beam at any given angular orientation relative to the section being examined and at any derived distance from its centre.

c) all the electronics necessary to drive the mechanics and to perform the data logging.

d) a computer for the data processing.

The most logical order in which to conduct the project would have been to consider the mathematics required for data processing (chapter 2) together with the various prediction problems (chapter 3) and then by taking these in conjunction with the desired results to arrive at a provisional specification for the equipment given in (a), (b) and (c) above. However, to fit in with such practical requirements as workshop time tables, delivery times and so on it was necessary to complete the design of the mechanics and detector system before the mathematics were completed.

This creates the situation of having to design the equipment without knowing what it is that one wants to design. As a result the basic philosophy behind the equipment design was:-

a) for those equipment parameters which required definite decisions (e.g. maximum section size to be accommodated) to arrive at some answer, if necessary by means of rough and ready reasoning.

b) for those parameters on which decisions could be deferred (e.g. linear and angular increments between samples of the projections) to design the equipment so that they remained variables which could be set at the time of scanning, hopefully in the light of knowledge and/or experience gained by that time.

In this chapter the tentative specifications (together with the reasoning leading to them), the realised specifications and a brief description of the overall system and its various components is presented.

·2 Specification of scanner

·21 Tentative Specifications

Two relationships were used to arrive at a tentative specification for the equipment, the first giving values for the count rate and scan time and the second for the angular accuracy.

Suppose the scanning is performed with the following parameters:-

the scanning speed $= v$

the collimater radius $= a$

the source-detector distance $= d$

and that a monoenergetic photon source radiating n photons/sec and a reconstruction cell size of Δx are used.

Then for $a << d$ the fractional solid angle subtended by the detector collimater is

$$\frac{\pi a^2}{4\pi d^2}$$

and the time for which each cell is viewed by the source/detector system is $\Delta x/v$ seconds. Thus the number of counts/cell when there is only air between source and detector (C) is given by

$$C = n \cdot \frac{\Delta x}{v} \cdot \frac{\pi a^2}{4\pi d^2} \quad \text{counts/cell} \qquad \cdot 1$$

and the number of counts/cell when tissue is placed between source and detector (C_t) is given by

$$C_t = C \exp\left\{ - \int_L f(\ell)\, d\ell \right\}$$

where $f(\ell)$ is the linear attenuation coefficient of the tissue along the path L (see F·1).

Consider C_t in two particular cases:-

1) $f(\ell) = f$, a constant, then $C_{t1} = C_{exp}(-f_o x')$ where x' is the width of the section ·2

2) $f(\ell)$ is a constant f_o on L except on an element of length Δx where it is increased by p% , then

$$C_{t2} = C \exp \{-f_o x' - f_o p\Delta x/100\}$$

$$= C_{t1} \exp \{-f_o p\Delta x/100\}$$

If this difference in the sections is to be detectable it must produce a statistically significant change in the number of counts/cell. If the number of counts/cell is C_{t1} then, assuming Poisson statistics the standard deviation in the number of counts/cell is $\sqrt{C_{t1}}$.

The change in the number of counts/cell due to the difference in the two sections is

$$\Delta C_{\perp} = C_{t1} - C_{t2} = C_{t1}\ (1 - \exp(-f_o p\Delta x/100)) \qquad \cdot 3$$

and if this is to be significant at the 3σ level then

$$\Delta C_{t1} \geq 4\sqrt{C_{t1}} \qquad \cdot 4$$

hence from E·3 and E·4

$$C_{t1} \geq \frac{9}{(1 - \exp(-f_o \Delta p\ x/100))^2}$$

and from E·2

$$C \geq \frac{9 \exp(f_o x')}{(1 - \exp(-f_o p\Delta x/100))^2} \qquad \cdot 5$$

and substituting E·1

$$\frac{na^2 \Delta x}{4\ d^2} = C \geq \frac{9 \exp(f_o x')}{(1 - \exp(-f_o p\Delta x/100))^2}$$

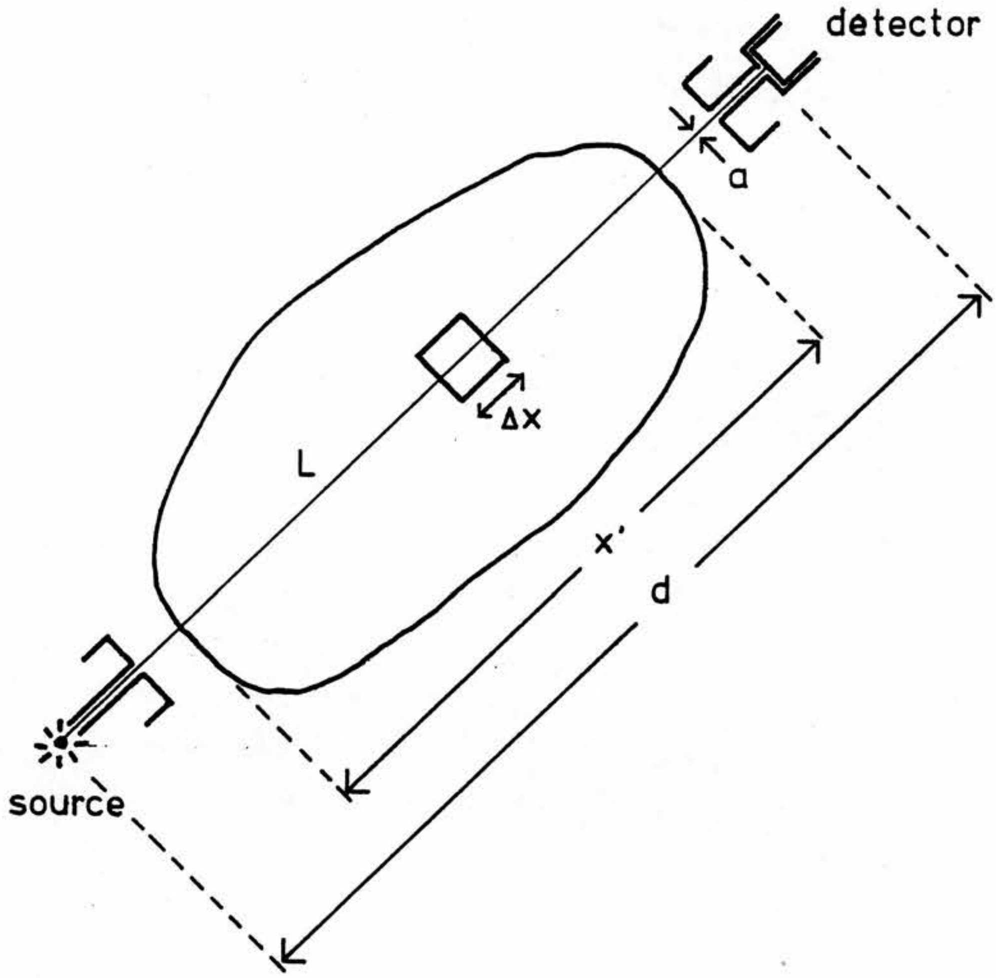


fig. 4·1

and dividing through by $\Delta x/v$ (the time/cell)

$$C' = \frac{na^2}{4d^2} \geq \frac{9v \exp(f_o x')}{\Delta x(1 - \exp(-f_o p\Delta x/100))^2}$$

where C' is the count rate at which the detector is working when there is no tissue between source and detector. If the maximum count rate of which the detector is capable is C'_{max} then

$$C'_{max} \geq \frac{na^2}{4d^2} \quad \frac{9v \exp(f_o x')}{\Delta x(1 - \exp(-f_o p\Delta x/100))^2} \qquad \cdot 6$$

This 'rough' relationship between the various parameters was used as the basis for giving a tentative specification for the equipment. It is considered 'rough' for the following reasons:-

a) the reasoning leading to E·4 assumes that a statistically significant difference between the two scans must be observed in a given traverse, but there is no justification for this. All that matters is that there should be a significant difference between the total information in one scan and the total information in the second scan.

b) even if the above assumption is taken as correct E·4 is a necessary condition not a sufficient one.

The second relationship used to arrive at a specification is that given in §1·6221 and §3·313 E·35

$$\Delta\phi\omega_c D < 1 \qquad \cdot 7$$

where

$\Delta\phi$ = the angular increment between traverses (in radians)

ω_c = spatial resolution

D = diameter of section.

Numerical values were arrived at by assuming that the scanner was required to detect changes of 10% (of the linear attenuation coefficient of water) over a 1 cm cube in a 30 cm diameter section using a 1 curie source of Cs^{137}. (See

Preface §·1 for more detailed discussion.)

Consider first the question of count rate and scan speed. To allow for various spatial resolutions it was considered desirable to be able to use a range of collimaters from $a = 0{\cdot}05$ cm to $a = 0{\cdot}5$ cm. C'_{max} is determined by $a = 0{\cdot}5$ cm, thus assuming a source/detector spacing of 40 cms and a 100% efficient detector, E·6 gives

$$C'_{max} \leq \frac{0{\cdot}89 \times 3{\cdot}7 \times 10^{10} \times 0{\cdot}5^2}{4 \times 40^2} \text{ counts/sec}$$

$$\simeq 1{\cdot}28 \times 10^6 \text{ counts/sec}$$

Similarly putting $p = 10$, $\Delta x = 1$, $f_o = 0{\cdot}085 \text{ cm}^{-1}$, $x' = 30$ then E·6 gives

$$v \leq 0{\cdot}799 \text{ cm/sec for } a = 0{\cdot}5 \text{ cm}$$

and

$$v \leq 7{\cdot}99 \times 10^{-3} \text{ cm/sec for } a = 0{\cdot}05 \text{ cm.}$$

Hence with no section present the number of counts/cell is

$$1{\cdot}28 \times 10^6/0{\cdot}799 \simeq 1{\cdot}6 \times 10^6$$

which requires a 21 bit counter. With 30 cms of water present this decreases to about $1{\cdot}25 \times 10^5$ counts/cell which requires 17 bits. With the equipment available the only practical method of data logging was to use a PDP-12 on line to the scanner and this computer has a 12 bit data word. These facts immediately determine the number of bits required in data transfer to the computer. As will be seen in §·33 1 bit is required to carry position information, thus if 12 bit data transfers were to be used the counts would have to be scaled by 2^{10} to accommodate a 21 bit count. Now consider the lower count of 17 bits, this has a variance of about 2^8, thus if scaling by 2^{10} were to be used then at lower count rates errors will be introduced which are several times the statistical errors. Thus the only possibility which does not seriously distort the data statistics is to use 24 bit data words.

If 40 traverses of 40 cms are used in a scan then the data collection time will be

$40 \times 40/0{\cdot}799$ secs $\simeq$ 33 mins for $a = 0{\cdot}5$ cms

and 56 hours for $a = 0{\cdot}05$ cms

Now consider the angular accuracy, from E·7 with $\omega_c = 1\ \text{cm}^{-1}$, $D = 30$ cm

$$\Delta\phi < \frac{360}{2\pi} \times \frac{1}{30}\ {}^{\circ} \simeq 2^{\circ}$$

and if 5% accuracy is required then the angular accuracy must be of the order of 6'.

Summarising these tentative specifications:-

source activity	1 curie C_s^{137}
collimater radius	0·05 cm - 0·5 cm
max. detector count rate	$1{\cdot}28 \times 10^6$ counts/sec
max. counts/cell	$1{\cdot}6 \times 10^6$ counts/cell
data word length	24 bits
angular accuracy	6'
linear accuracy	<< 0·05 cms
speed	8×10^{-3} cm/sec to 0·8 cm/sec

·22 Specifications realised in practice

source activity	as specified
collimater radius	" "
max. detector count rate	better than 450kHz with no detectable non linearity*
max. counts/cell	$\simeq 8{\cdot}38 \times 10^6$
data word length	as specified
angular accuracy	better than 2'
linear accuracy	better than $0{\cdot}5 \times 10^{-3}$ inches†
speed	$1{\cdot}27 \times 10^{-3}$ cm/sec - 2·54 cm/sec continuously variable

*This figure is derived by using the 1 curie source and the 0·5 cm collimater and is thus the maximum count at which the detector will ever have to work (see §·38)
†See §·31 for exact meaning.

·3 Overall system description

The scanner consisted of three sections (see F·2):-

a) the mechanical scanner providing translation and rotation, the detector, various position sensors and a stepping motor.

b) the scanner electronics which included:- an H.T. unit, amplifiers, discriminator, NlM-TTL interface and local count rate display for the detector system; a variable frequency pulse unit and drive unit for stepping motor; a line driver unit for communicating with the computer interface; a control unit which co-ordinated the functioning of the entire scanner; and an alarm unit to inform the operator of fault conditions on the remotely located computer.

c) a PDP-12 computer and suitable interface.

·31 Scanner mechanics

The mechanical section of the scanner is illustrated in F·3 to F·4 The section to be scanned is placed on a standard commercial milling table (accurate to 2') and rotated manually relative to the source/detector assembly.

The source and detector are both housed in lead shielding and supported on a U shaped frame whose weight is taken by four linear bearings running on two steel rods. This assembly is moved along the bed of the scanner by a stepping motor attached to the centrally positioned drive shaft. This shaft is connected to the U frame by two ball bearing screws mounted back to back one of them is rigidly mounted on the frame, the other is adjusted to take up any slack in the bearings. When correctly adjusted the maximum movement of the base of the U frame relative to the scanner base plate due to play in the bearings was better than 0·0005". The maximum diameter of section that can be accommodated is 35 cm and the traverse length is 40 cms.

·32 Drive circuitary

The drive circuitry is made up of a purpose built T.T.L. pulse generator, a commercially available drive unit (Unimatic USD 853) and a stepping motor

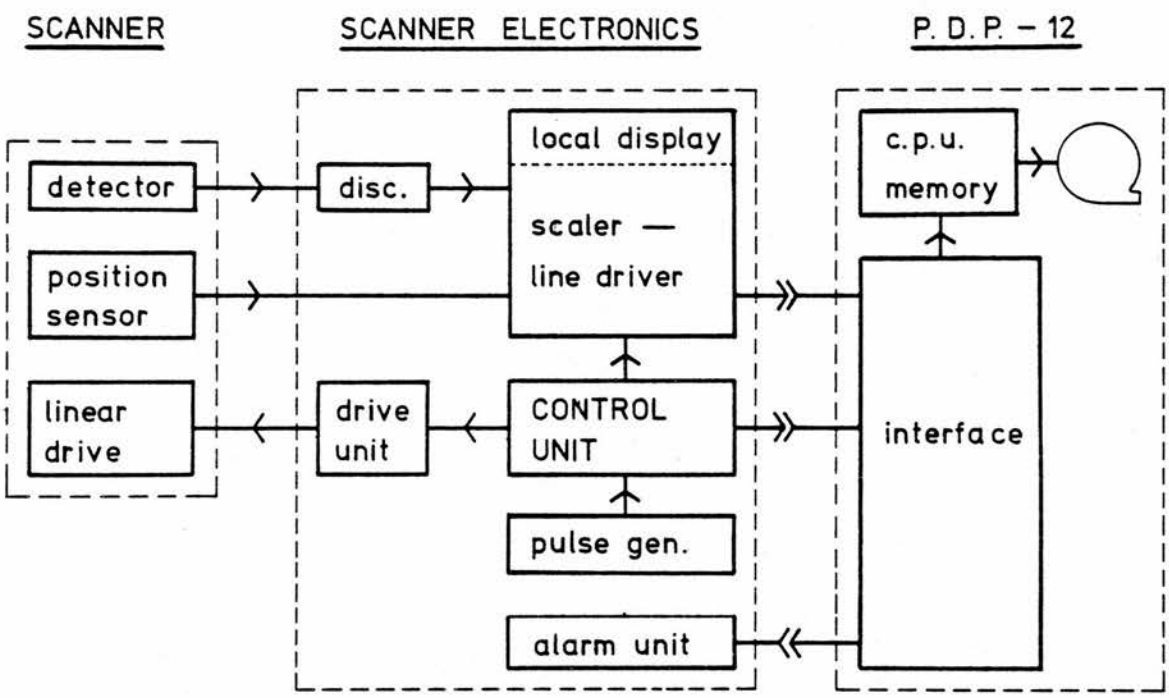


fig. 4·2 : Scanner structure

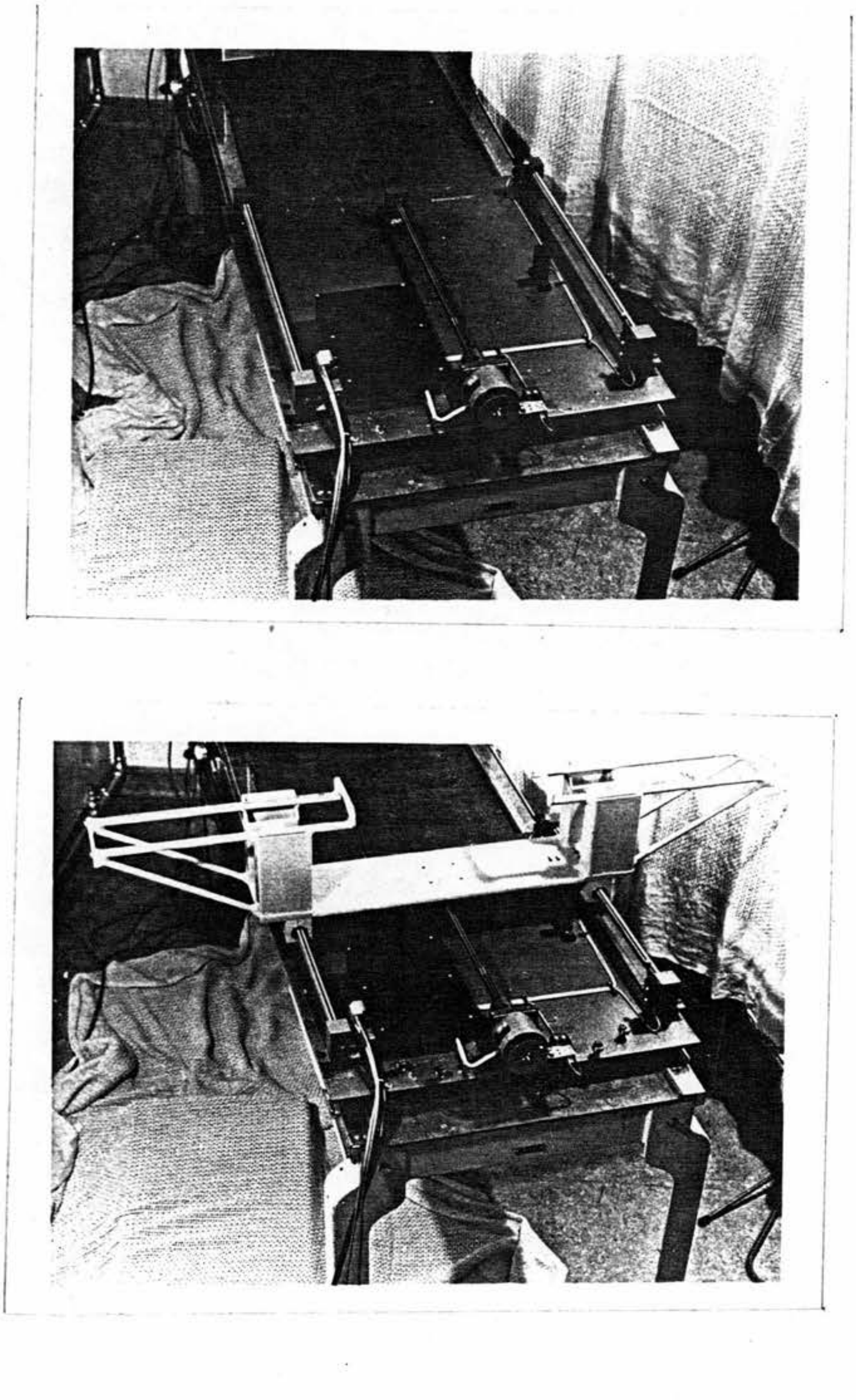


fig. 4·3 : The scanner

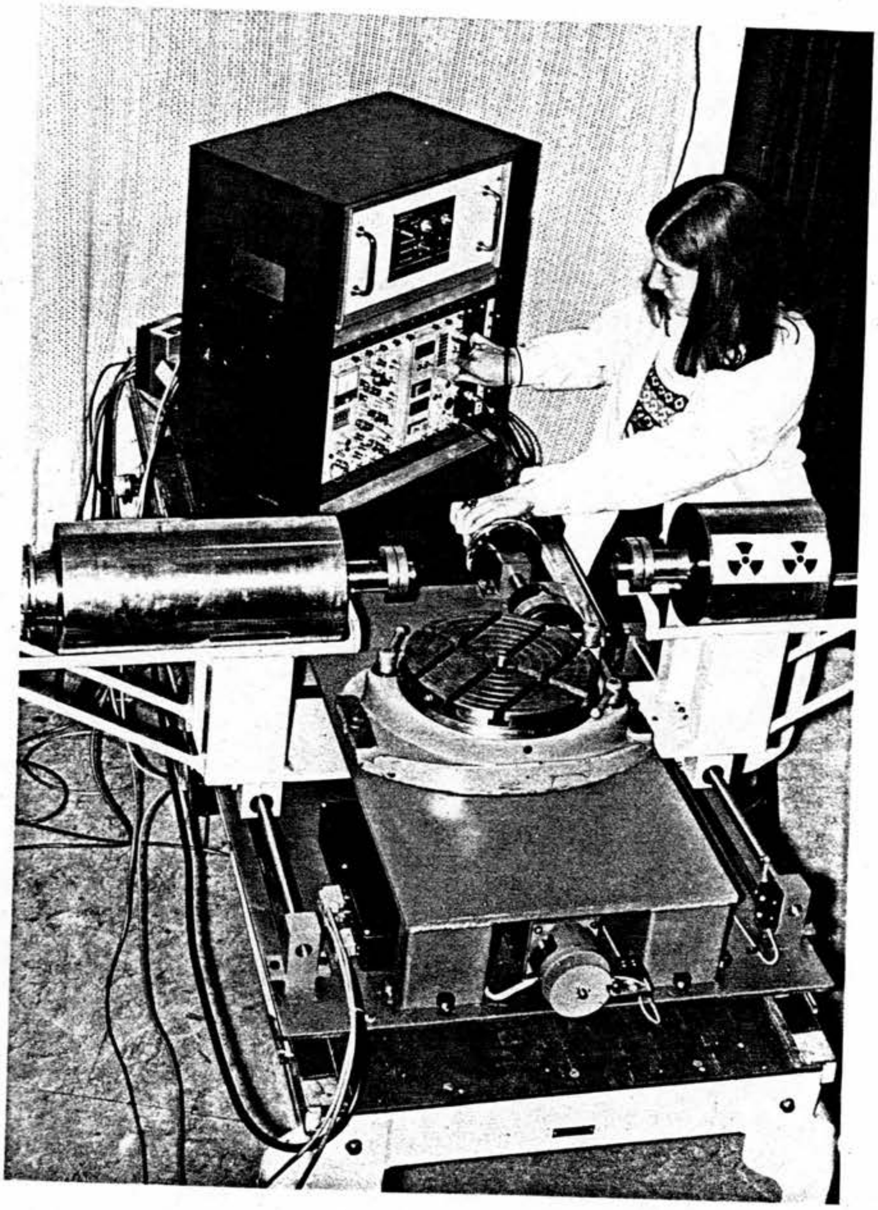


fig. 4·4 : The scanner

(Unimatic 20-3424 D200-E1·8B). The stepping motor has a maximum torque of 80 oz in and is used in 8 step mode giving 400 steps/revolution. Since the lead screw has a pitch of 5 threads per inch, 1 step of the motor gives a translation of 0·0005". The required speeds for traverse are 8×10^{-3} cm/sec - 0·8 cm/sec or 6·3 steps/sec - 630 steps/sec. The range of frequencies delivered by the pulse generator is 1 Hz to 2kHz, thus amply covering the required range. However, with the speeds required and the moment of inertia of the screw it is not possible to start the motor at 600 steps/sec, the pulse generator therefore incorporates ramping to accelerate the traverse in a controlled manner and no data collection is performed until full speed is reached.

The output of the pulse generator is passed to the control unit and, with the addition of clockwise/anticlockwise information, onto the drive unit. The same pulses are also passed from the control unit to the 'Scalar, Counter, Line driver' unit where they are scaled down by a factor of $2^n (n \geq 4)$ and are used to define the end of each cell during which counting of scintillation pulses takes place. These end of cell pulses are aligned by the mechanics and electronics so that the centre of rotation of the milling table coincides with the junction between two cells.

·33 Position information

The position information for monitoring the position of the U-frame during the traverse is derived from two sources:-

a) Two light switches are used to give an output pulse as the U-frame passes the centre of rotation of the table. One monitors the angular position of the motor and lead screw and gives an output pulse whose length is approximately equivalent to two steps of the motor and which occurs at a fixed point once in every revolution of the motor (see F·5). The other monitors the U-frame position by a reflecting surface on the underside of the frame passing over a phototransistor/ l.e.d. assembly mounted on the base plate of the scanner and gives a pulse at just one point in each traverse of the scan.

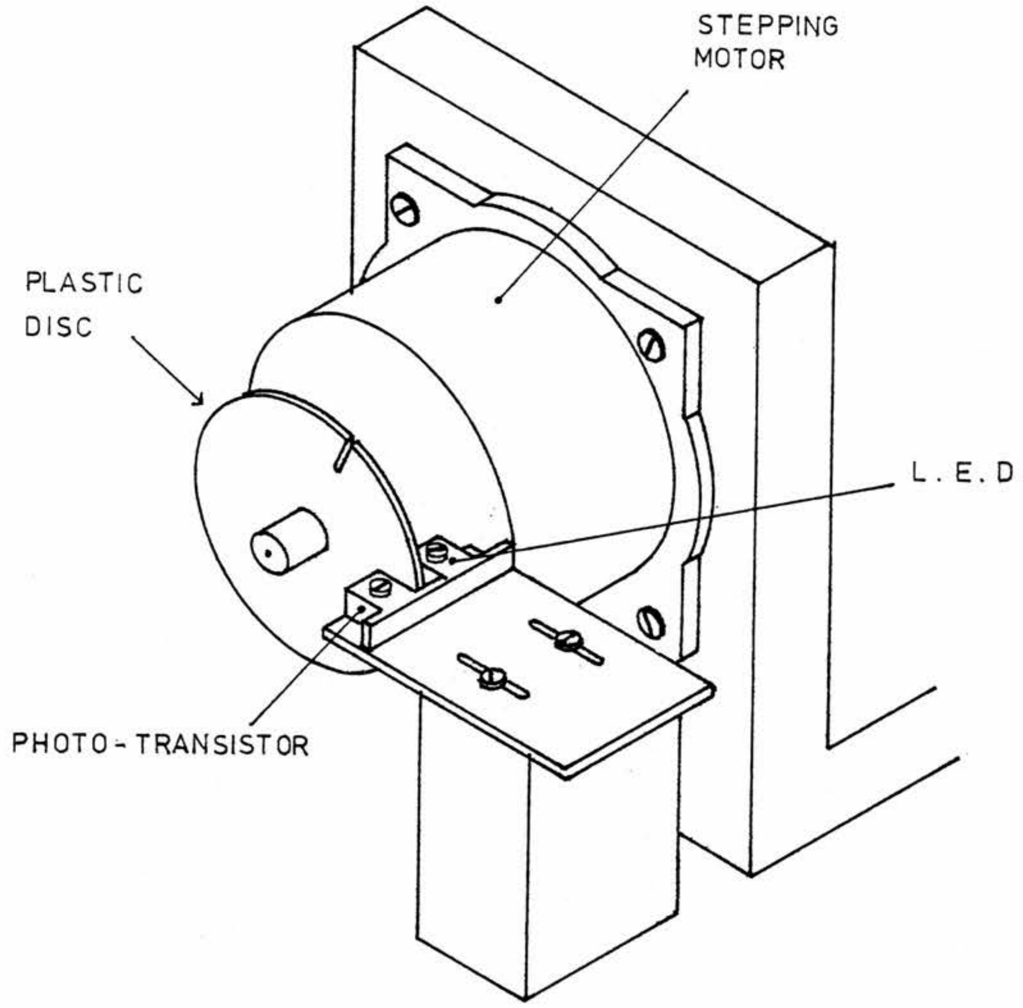


fig. 4·5 : Angular light switch

This pulse has a length corresponding to approximately 0·1" or 200 steps of the U-frame motion (see F·6). These two pulses are gated together to give a single pulse whose position is adjusted (by moving the light switches) to correspond to the centre of rotation of the table.

b) As noted in §·32 the stepping motor pulses are divided down and used to define the ends of each cell.

These two sets of pulses are synchronised so that the single pulse marking the centre of rotation corresponds to an end of cell pulse, thus defining the absolute position of every end of cell pulse in the traverse. This information is passed to the computer interface via the control unit. The particular end of cell pulse corresponding to the centre of rotation is identified to the computer interface by using this same pulse to change the state of a flag whose output is available to the interface. Thus for all cells on one end of the traverse the flag is high, and for those on the other end the output is low. The state of this flag is transferred into the computer as the most significant bit of the 24 bit data word containing the accumulated count from the detector (see §·35).

·34 Data path

The data path along which the output of the detector is passed to the computer contains five components. The low level scintillation pulses from the detector are fed to an NE4634 fast amplifier and thence to an NE4635 fast discriminator set to pick off only the photo peak of the spectrum. The output of this unit is a current pulse (in accordance with the fast NlM specs) and is therefore put through a NlM - T.T.L. interface before going to the 'Scalar, Counter, Line-driver unit'.

At this point the data is used in two ways. One copy of the data is forwarded by line drivers to the remote computer interface. These pulses are random T.T.L. compatible pulses corresponding to photopeak scintillations. The second copy of the data is input to a local 24 bit counter and binary

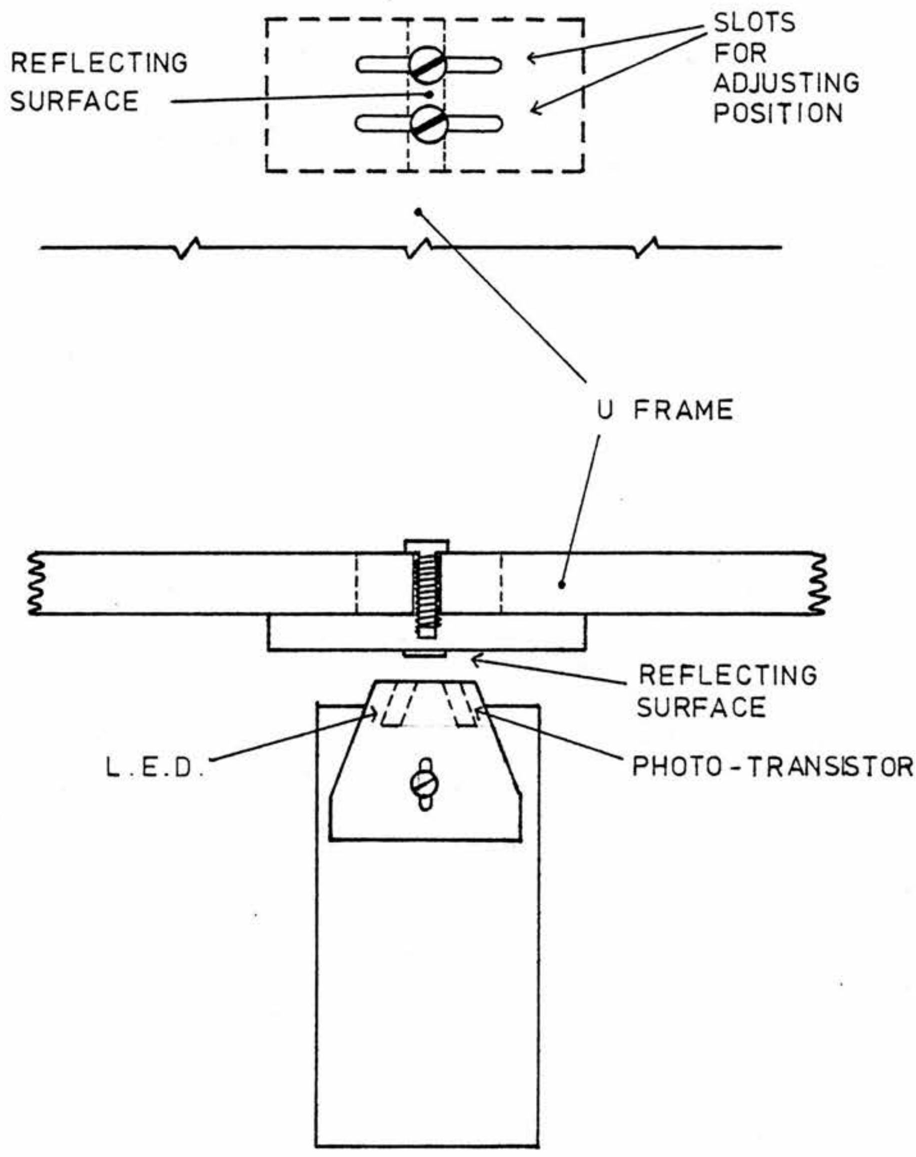


fig. 4·6 : Linear light switch

display so that the scanner operator has a visual display of how close he is to overflowing the capacity of the 24 bit data words.

·35 Computer interface

The computer interface is mounted in the computer and remotely located from the scanner. The interface consists of two 24 bit counters and the necessary logic to multiplex between them. The counters are arranged so that one is counting data coming from the scanner while the other is being read and cleared by the computer, changeover from one state to the other is controlled by the end of cell signals from the scanner. Since the computer is based on a 12 bit word, two instructions are required to read the counter; one to read the 12 most significant bits and one to read the 12 least significant bits. Only the 23 least significant bits of the counters are transferred to the computer, the 24th bit from both counters is fed back to the scanner as an overflow indicator and in place of it the output of the position flag is read in as part of the data word (see §·33). The interface is designed to be operated either by Program Data Transfer or by Program Interupt.

The software driving the interface, after requesting data concerning the setting of various controls on the scanner, simply initialises the interface and then dumps data from the interface into memory. At the end of the traverse the core data buffer is copied onto LINC tape and the software awaits the start of the next traverse unless told otherwise.

·36 Control unit

The control unit has the task of orchestrating the entire functioning of the system. Once various parameters such as scan speed, cell size, etc. have been set on the scanner electronics and the data logging program set running, the operator communicates with the system through this unit. The only other points of control being the manual setting of the milling table and the computer teletype on which the end of a scan sequence is indicated to the software. It is through the control unit that the operator may control such functions as move left/stop/move right, inhibit all data transfer to computer, store or erase the last traverse, etc, as well as monitoring such fault conditions as counter overflow, U frame hitting end stops, and so on.

·37 Mechanical construction

The scanner electronics is based on a standard N.I.M. bin (NE4601) with the NE4655 low voltage power supply and NE4646 high voltage supply. All the purpose built modulus (NIM-TTL interface, Pulse generator, Control unit, Alarm unit, Scalar/Counter/Line-driver) are housed in single or double width NIM units, with T.T.L. logic mounted on glass fibre boards and connected by wire wrapping as the basic form of construction.

·38 The detector system

It was required to work at count rates of around 1-10MHz (§·21) and it was decided to retain the possibility of energy discrimination. After considering various possibilities a NaI(Th) and photomultiplier detector system was decided on. With this detector type there are two major design constraints:-

a) anode current must be limited to give stable operation of the P.M. tube, and at this energy (660 kev) and count rate this means a low gain tube.
b) if one is to use pulse height discrimination at 1MHz with random pulses then the pulses must be less than 100 ns wide to avoid pulse pile up. Since NaI(Th) pulses are about 1μs this means that pulse shaping must be used

PRELIMINARY WORK was done using an aged 2" × 2" crystal and an EM1 6097B P.M. tube (11 dynodes) in a grounded anode configuration (see F·7a). The two outlets being used to assess both the anode current at different count rates (see F·7b) and the effectiveness of the bridge-T network suggested by Amsel et al (Amsel et al, 1969) for pulse shaping (see F·8a and F·8b). Extrapolation of

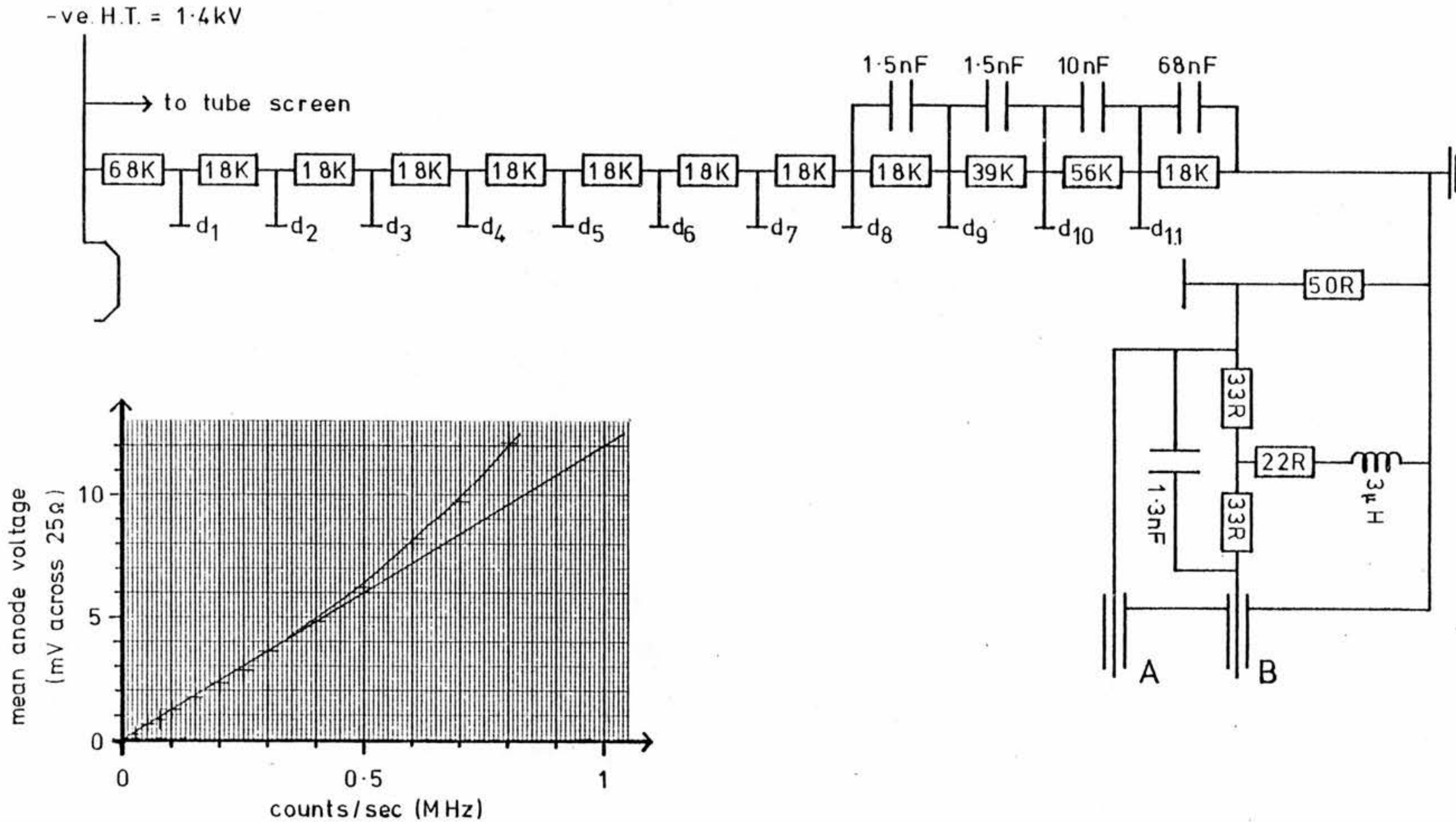
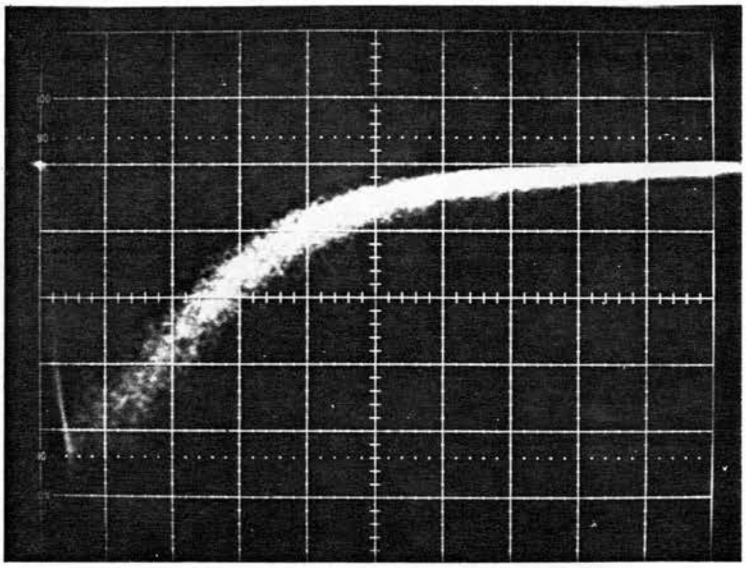


fig. 4·7 (b)

fig. 4·7 (a)

fig. 4·7 : Preliminary work

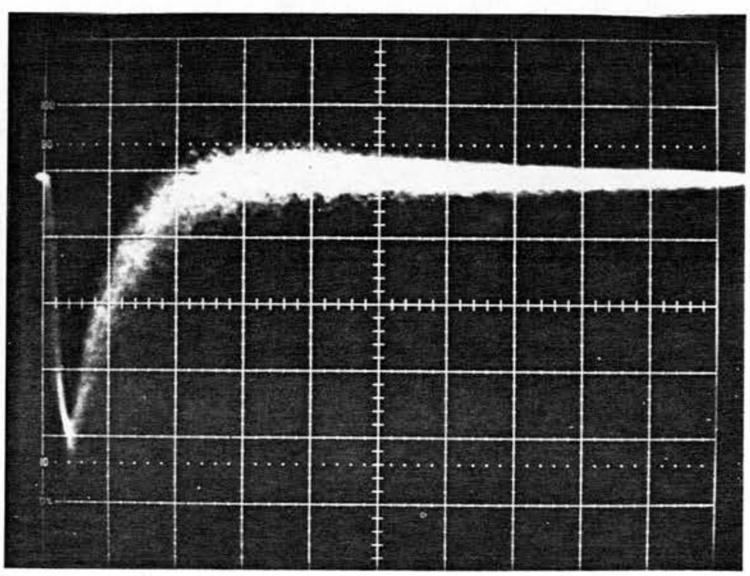


hor 0·1μs/div
vert 5mV/div

output A of fig. 4·7a

anode load = 50/3 Ω

fig. 4·8a



hor 0·1μs/div
vert 5mV/div

output B of fig. 4·7a

anode load = 25Ω

fig. 4·8b

fig. 4·8 : Preliminary work

the linear region of the graph of anode current against count rate (F·7b) suggests that (if it were not for pulse pile up) the current at 1MHz would be about 0·5 mA. For stable operation an anode current of less than 10μA is required thus the gain must be reduced by at least 50. Inspection of the signals from the two outputs of the test set up showed that the signal can be reduced to about 200 ns without undue effort.

FINAL DESIGN - INTRODUCTION. As a result of the preliminary work it was decided that a detector should be built using a new 2" × 2"NaI(Th) crystal with an EMI 9637B PM tube (6 dynodes). The tube delivered had an overall sensitivity of 1A/L at 860V. Because of the drastic reduction in detector output (a factor of 10^2 - 10^3 down) it was also decided to increase the anode load resistor to at least 500R and to incorporate a × 10 head amplifier (NE 5297G) so that a reasonable signal level was present at the output. The use of the head amplifier also has the bonus of making impedance matching to a 50R cable easy.

The procedure for designing the new detector was as follows:-

a) copy the standard EMI dynode chain used to calibrate the tube and operate the tube at the specified voltage (in this case 860V) thus giving a known overall gain (in this case 1A/L). Using the known sensitivity one can then calculate the light output of the crystal. (N.B. in this case the signal levels are so low that it is necessary to use the head amplifier in order to see the output at all. This means that in addition to the above the gain of the head amplifier must also be calibrated.)

b) use the figures for the crystal light output to calculate the overall gain required to give an anode current of ≈10μA. Use the tube characteristics to calculate the overall voltage required to give this gain. For good linearity the dynode current should be about 100 times the anode current (i.e. ≈ 1mA), this fact together with the new overall voltage determines the dynode resistors.

c) check that with the new dynode chain design none of the absolute maximum ratings of the tube are exceeded

d) design the new pulse shaping network

e) check the final performance for linearity of count rate and possible distortion of spectra at high count rates

AMPLIFIER GAIN CALIBRATION. The amplifier uses the 500R resistor both as the anode load resistor for the PM tube and as an integral part of the bias network for the input transistor. It is therefore necessary to ensure that the amplifier is driven from a high impedance; hence the 27K resistor. Both input and output were measured with a Tektronix oscilloscope using ×10 probes The gain of the amplifier was found to be about 8 (see F·9a).

The INITIAL TEST CHAIN is shown in F·9b. The r"C combination was found desirable to suppress 40kHz spikes arising from within the HT unit. The other components (r' and r) are assigned so that with $V_{k-a} \simeq 860V$ (the test voltage for this tube) the dynode current is ≃ 1mA and $V_{k-d1} \simeq 150V$ (the -r-r-2r-r chain being the standard EMl test chain). With this circuit $V_{k-a} \simeq 860V$ when the overall voltage is set at -1060V.

CALIBRATION OF CRYSTAL LIGHT OUTPUT. With an overall voltage of -1060V, $V_{k-a} \simeq 860V$ giving an overall sensitivity of 1A/L (from the calibrated values supplied with this tube). Working at low count rates and examining only the photopeak of the spectrum with a suitable scope at the point indicated in F·10:-

	the peak photopeak output voltage	= 45 mV
∴	the peak photopeak input voltage	= 45/8 mV
∴	the peak photopeak anode current	$= \frac{45}{8\times500}$ mA
		= 11·2 μA
	overall sensitivity (at 860V)	= 1 A/L
∴	peak photopeak illumination	= 11·2μL

Note also that:-

cathode sensitivity (from test ticket)	= 55μA/L
hence peak photopeak cathode current	= 11·2 × 55 nA
	= 0·6 nA

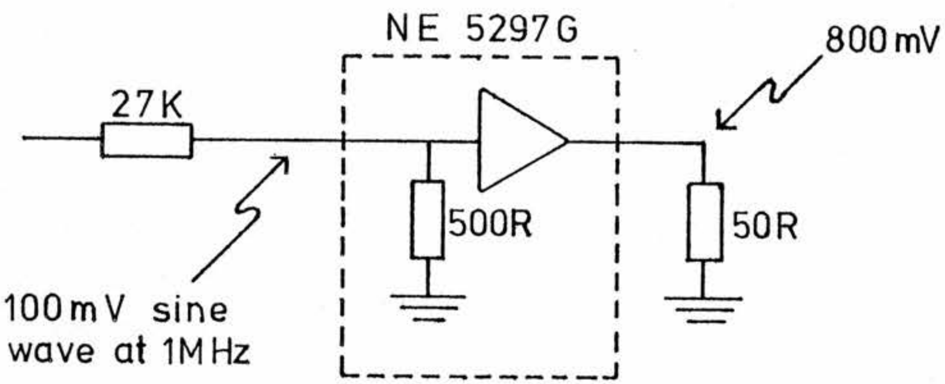
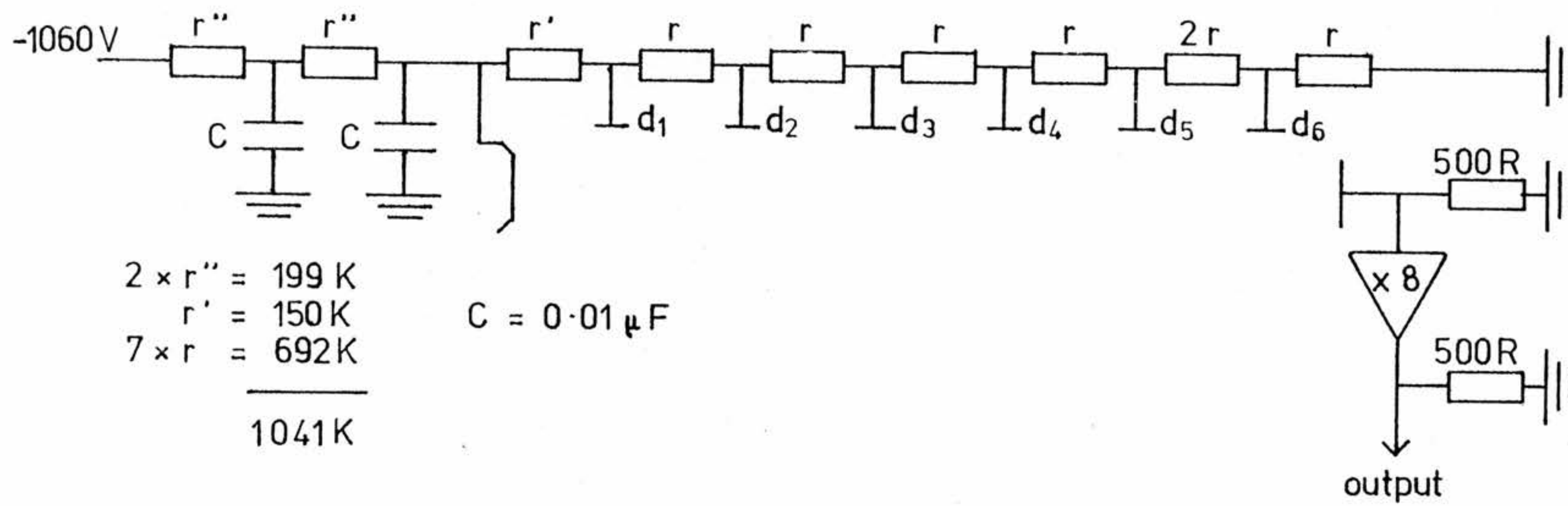


fig. 4·9a : Amplifier gain measurement

fig. 4·9b : Initial test chain



and this is well within the absolute maximum value of 0·3μA for this type of PM tube.

DESIGN OF FINAL DYNODE CHAIN. It is desired to adjust the various parameters so that the R.M.S. anode current is about 10μA and the dynode current about 1mA. As it happens the test chain has given a peak photopeak current of 11·2 μA. Thus at count rates of 1MHz one may expect that the R.M.S. anode current will approach 10μA and the test chain is seen to be almost the required final design.

However, one modification was made. It is desirable to be free to adjust the gain by altering the overall voltage. Since the detector is operating under conditions where pulse length is critical it is preferable to make sure that $V_{k-d_1} \geq 150V$ at all times. For this reason r' is increased to 199K and the overall voltage to about 1090V thus raising V_{k-d_1} to 200V and allowing one to increase or decrease the overall voltage (subject to not exceeding the absolute maximum ratings for the tube).

ABSOLUTE MAXIMUM RATINGS. There are five ratings to be considered for this type of tube

a) $V_{k-d_1} < 300V$

b) $V_{k-a} < 2000V$

c) Overall sensitivity<5A/L

d) $I_A < 1mA$

e) $I_k < 0{\cdot}3\mu A$

The overall voltage is constrained by the facts that $V_{k-d_1} > 150V$ for good pulse shape and that $V_{k-d_1} < 300V$ from (a) above. Hence

$$\frac{150}{200} \times 1090 < \text{Overall voltage} < \frac{300}{200} \times 1090$$

$$800 \quad \text{Overall voltage} < 1600$$

With this restriction clearly requirements (a) and (b) are both obeyed. Further from the basic design $I_A \leq 10\mu A$ so restriction (d) is obeyed.

It was shown earlier in this section that the peak photopeak cathode current was about 0.6nA so restriction (e) is obeyed and it remains only to check (c).

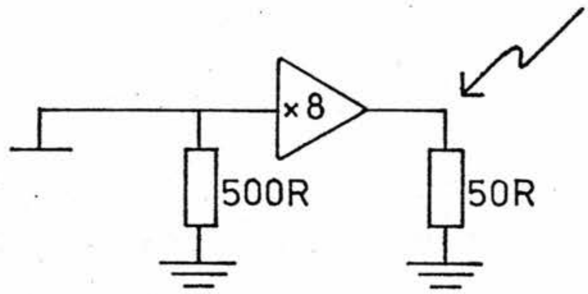
Now (from F·10) the peak photopeak output voltage ≃ 45 mV when the overall sensitivity is 1A/L. Hence if (at some other overall voltage) the peak photopeak output voltage is v mV the overall sensitivity will be v/45 A/L. Measurement of v for various overall voltages gives:-

Overall voltage (V)	Peak photopeak output voltage (mV)	Overall sensitivity (A/L)
600	5	0·11
700	10	0·22
800	17·5	0·39
900	27·5	0·61
1000	40	0·89
1100	55	1·22
1200	70	1·6
1300	90	2·0
1400	120	2·7
1500	150	3·3
1600	175	3·9

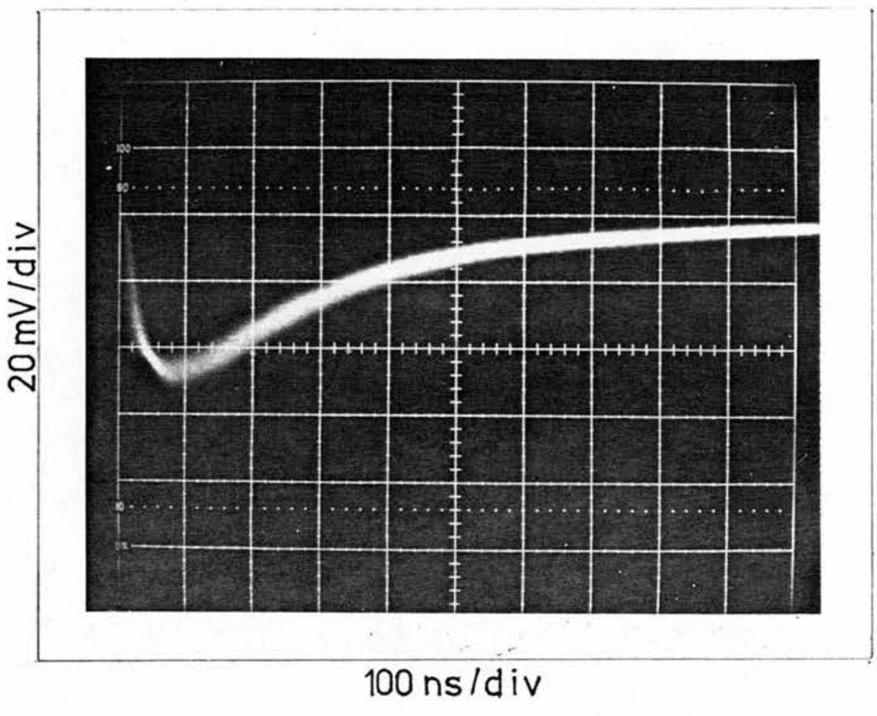
(see also F·11). Thus with the overall voltage <1600V as implied by (a) requirement (c) is also obeyed.

DESIGN OF PULSE SHAPING NETWORK. The network used is that suggested by Amsel et al (Amsel et al, 1969) and the discussion following assumes that this paper is already familiar. The basic concept behind this application of the bridge-T network is that the NaI(Th) scintillation pulse can be roughly approximated by a decaying exponential and since the network can be used to substitute an exponential decay of shorter time constant, the original pulse can be shortened.

The anode circuit of the detector including load resistor, pulse shaping network, and amplifier is shown in F·12 together with the formulae



anode circuit



anode output

fig. 4·10 : Output from detector with test chain

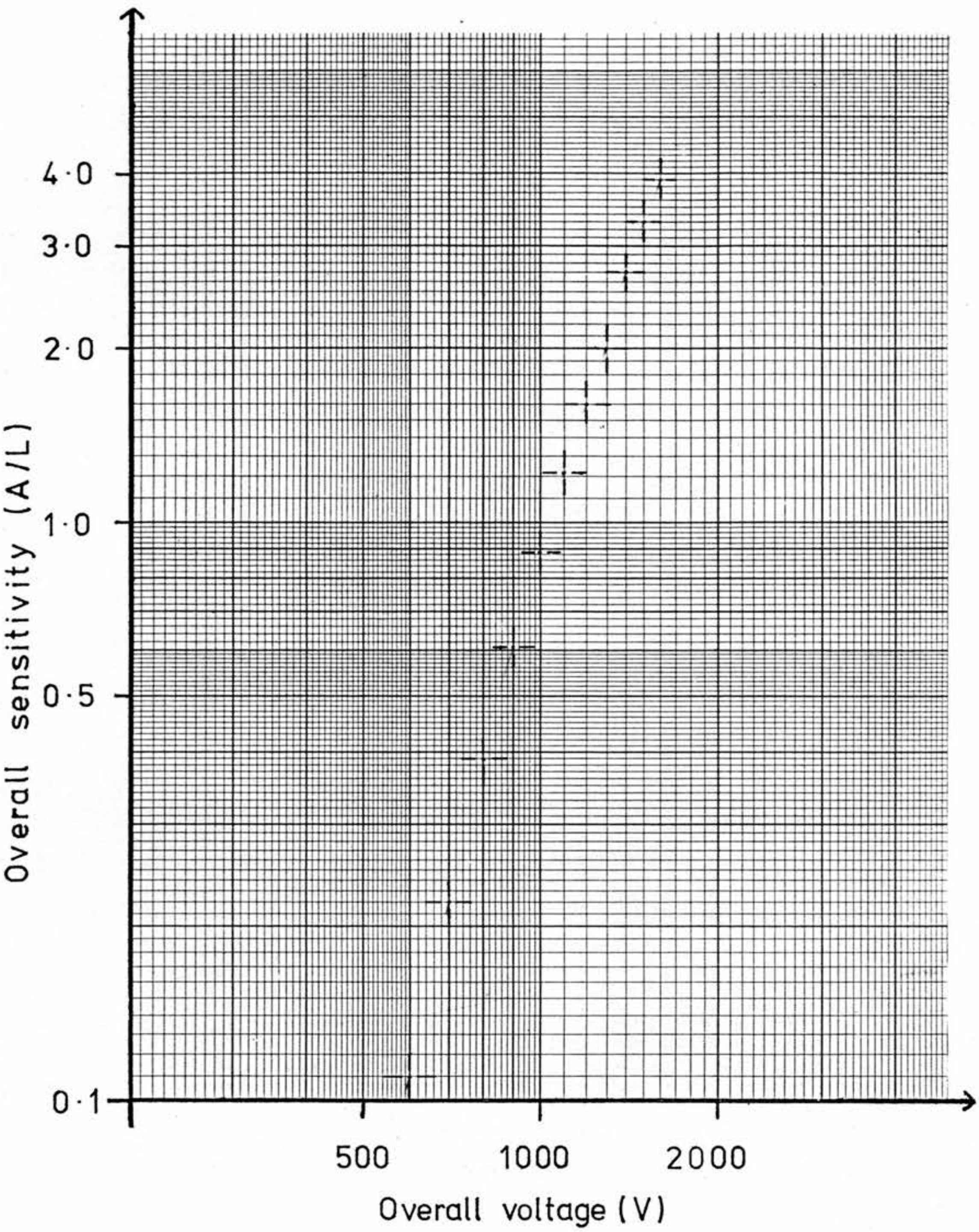
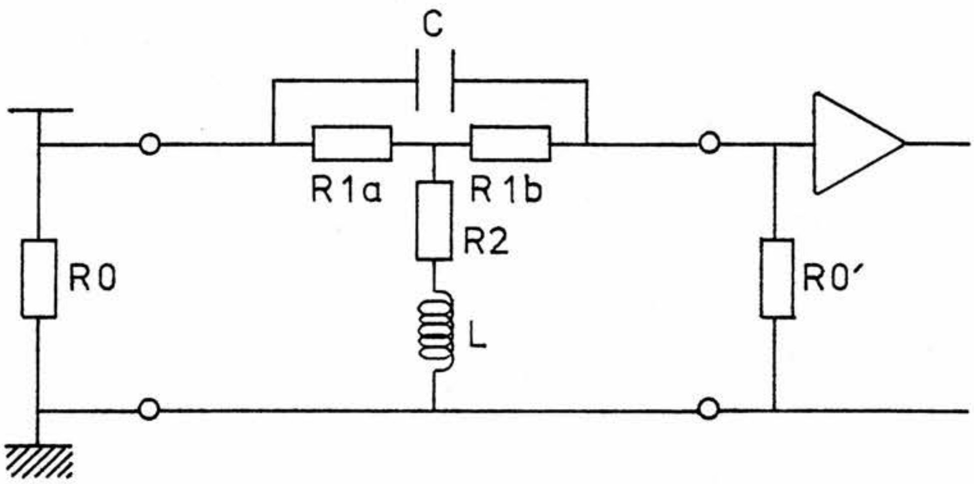


fig. 4·11 : Sensitivity vs. voltage



$$R1a = R1b = \frac{\gamma - 1}{\gamma + 1} R0$$

$$R2 = \frac{2\gamma}{\gamma^2 - 1} R0$$

$$C = \frac{\tau}{R0(\gamma - 1)}$$

$$L = \frac{\tau R0}{\gamma - 1}$$

R0 characteristic impedance

γ shortening ratio

τ time constant for unmodified pulse

fig. 4·12 : The detector anode circuit

necessary to calculate the different component values in terms of the three fundamentals:-

R_O the characteristic impedance

τ the decay time constant for the unmodified pulse, about 250 ns

γ the time constant for the unmodified pulse/the time constant for the modified pulse

N.B. the formulae for C differs from that in Amsel et al which is in error.

As noted earlier, the head amplifier uses a 500R resistor as both anode load resistor and as part of the biasing for the input transistor. By changing the resistor (R_O') to 1K and setting the characteristic impedance of the network to 1K the correct biasing conditions are maintained and all the required impedances properly matched.

The unmodified pulse has length of about 1μs and it is desired to reduce this to 100ns. Thus a network with $\gamma = 10$ is appropriate. With τ, R_O and γ thus determined the network is given by:-

a) $R_O = 1K$ whence $R1a = R1b = 820R$

$\gamma = 10$ $R2 = 180R + 22R$

$\tau = 250$ ns $C = 35pF$ trimmer

$L = 27\mu H$

An oscillogram of this output is given in F·13. A second set of components was tried with $\gamma = 5{\cdot}6$

b) $R_O = 1K$ $R1a = R1b = 680R + 18R$

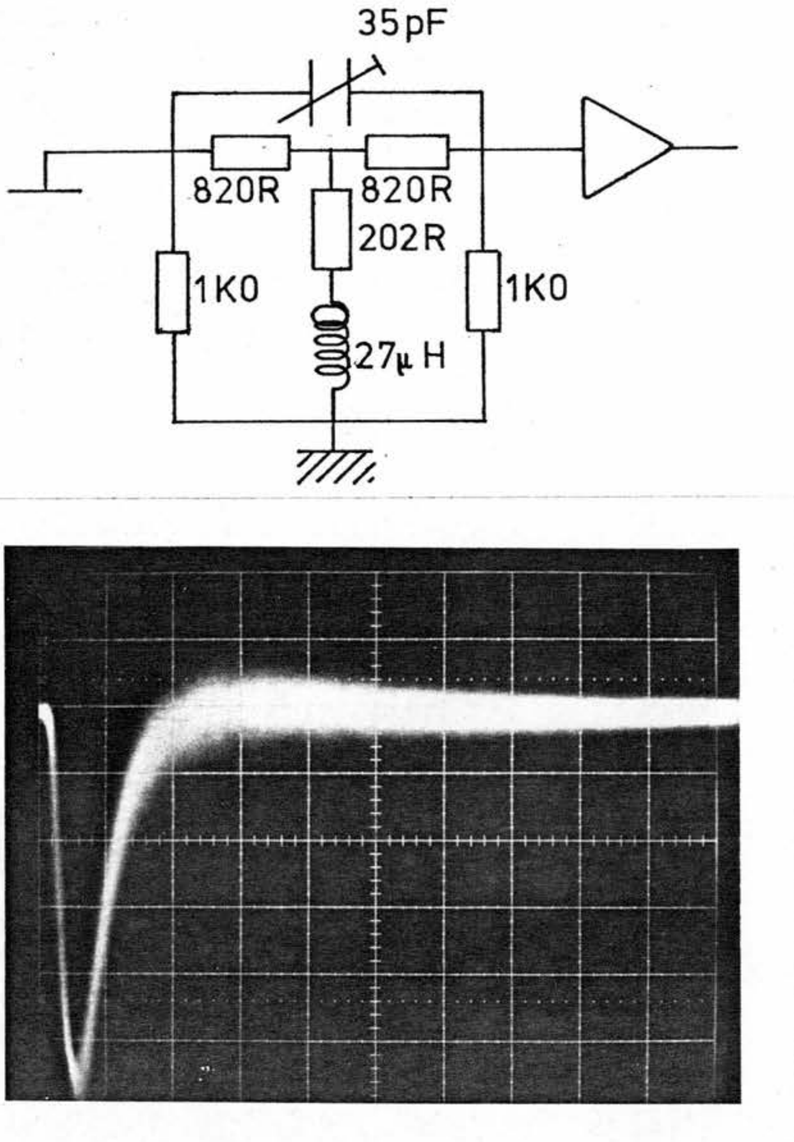
$\gamma = 5{\cdot}6$ $R2 = 330R + 39R$

$\tau = 250$ ns $C = 33pF + 35pF$ trimmer

$L = 27\mu H + 27\mu H$

An oscillogram of this output is given in F·14. Note that while this network gives a welcome improvement in pulse height (the reason for which is as suggested in Amsel et al) it is at the expense of pulse length.

It was then found (by accident) that if the resister R_O was omitted in filter (a) a further improvement in pulse height was possible with no visable deterioration in pulse shape (see F·15). This arrangement was therefore adopted.



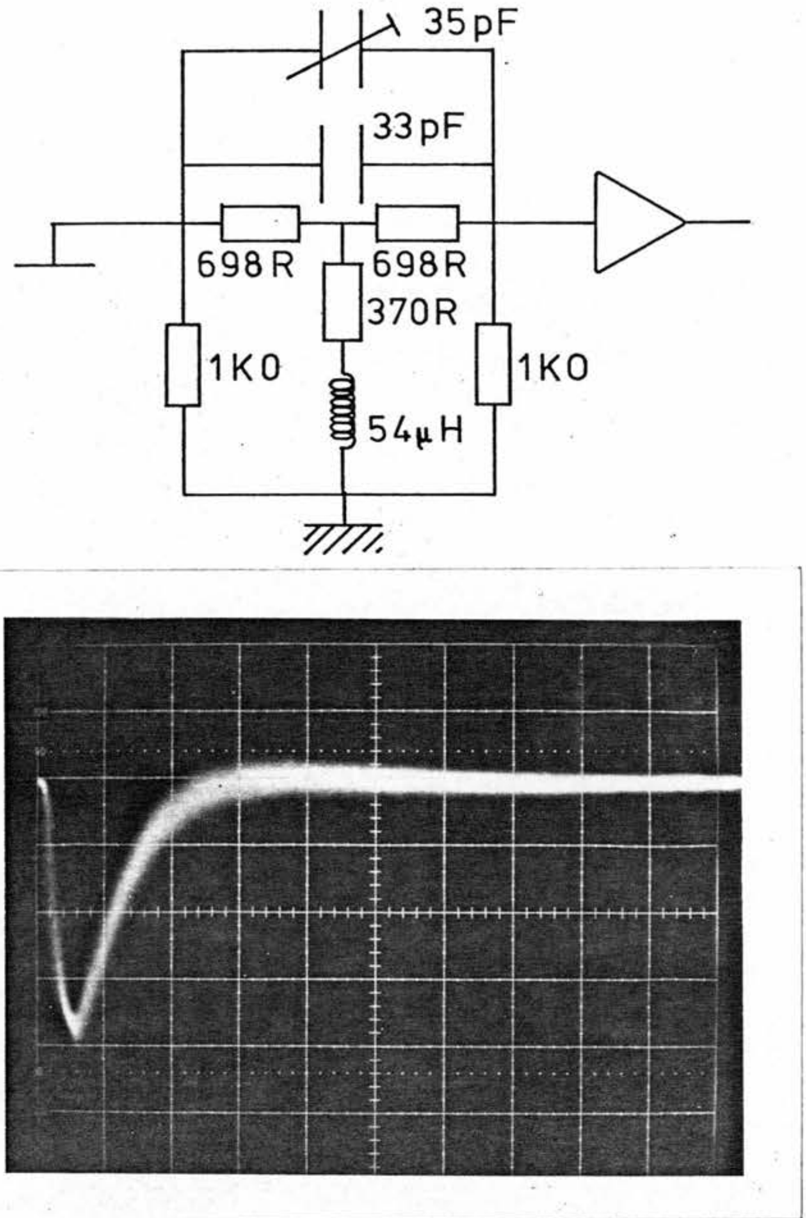
vert 5 mV/div

hor 100 ns/div

pulse length 200 ns

pulse height 27 mV

fig. 4·13 : Anode circuit and output, $\gamma = 10$



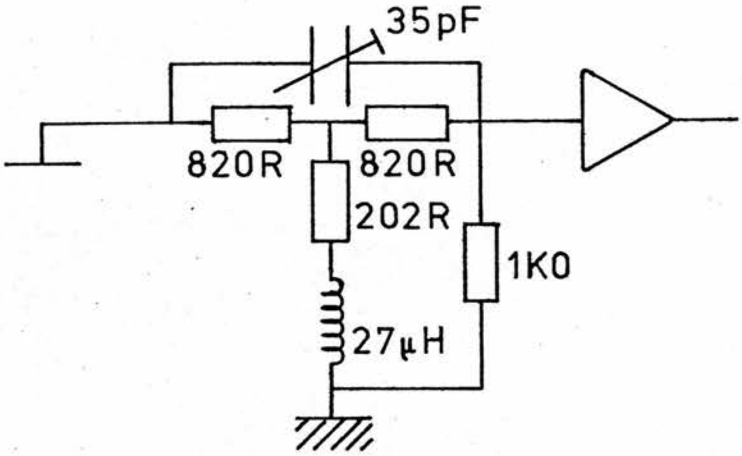
vert 10mV/div

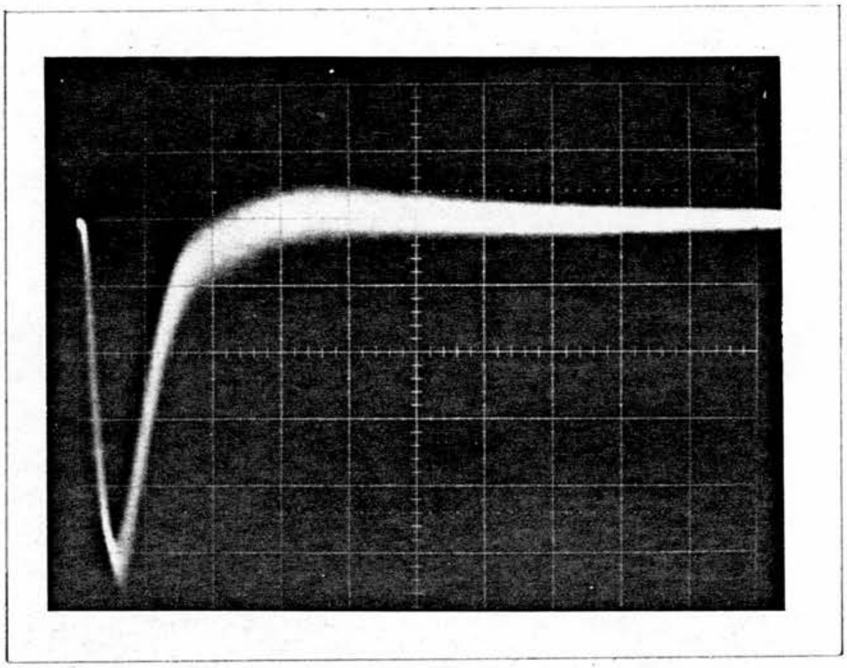
hor 100ns/div

pulse length 250ns

pulse height 37mV

fig. 4·14 : Anode circuit and output, $\gamma = 5{\cdot}6$





vert 10 mV/div pulse length 200 ns
hor 100 ns/div pulse height 50 mV

fig. 4·15 : Anode circuit and output pulse of final design.

The FINAL PERFORMANCE of the detector was evaluated in two ways:-

a) linearity: The detector was connected in series with its amplifier and discriminator, and a timer/counter (Bradley type 187). The discriminator was set so that the counter collected counts from the photopeak only of 1 Ci of Cs^{137} (the source with which it was subsequently to be used). Various combinations of brass blocks (with a cross section large compared to that of the beam) were placed directly in front of the source collimater to give different count rates. The results are given in F·16 and F·17 together with a drawing of the physical set up and a graph of the log(count rate) against attenuator thickness (see F·18 and F·19). In all cases the errors in count rate and thickness are too small to be plotted (see F·17). The results were felt to be satisfactory.

b) spectrum shape: The detector was connected in series with its amplifier and a spectrum analyser (TMC Gammascope MkII Model 102) and spectra taken with and without pulse shaping at different count rates. The physical set up was identical to that used for the linearity check except that no brass blocks were used and the count rate was varied by using detector collimaters of 1,2,3,5 and 10mm diameter (all 15 cm long). When used with pulse shaping the anode circuit of the detector was as given in F·15. When used without pulse shaping the anode circuit of the detector was as given in F·10. One problem encountered was that no sufficiently fast spectrum analyser was available, the absolute shape of these spectra is therefore questionable. However, in this case the interest is basically in possible distortion of the spectra due to pulse pile up rather than in the shape of the spectra themselves and provided the results are examined with this in mind they remain, in the writer's opinion, worthwhile. In the absence of pulse shaping the effects of the pulse pile up are discernable in both 5 and 10 mm spectra (see F·20b and F·20c:) and the improvement is obvious when pulse shaping is added (F·21) (N.B. With 10 mm collimaters the photopeak count rate is about 210KHz, so that the count rate for the entire spectrum is about 450KHz.)

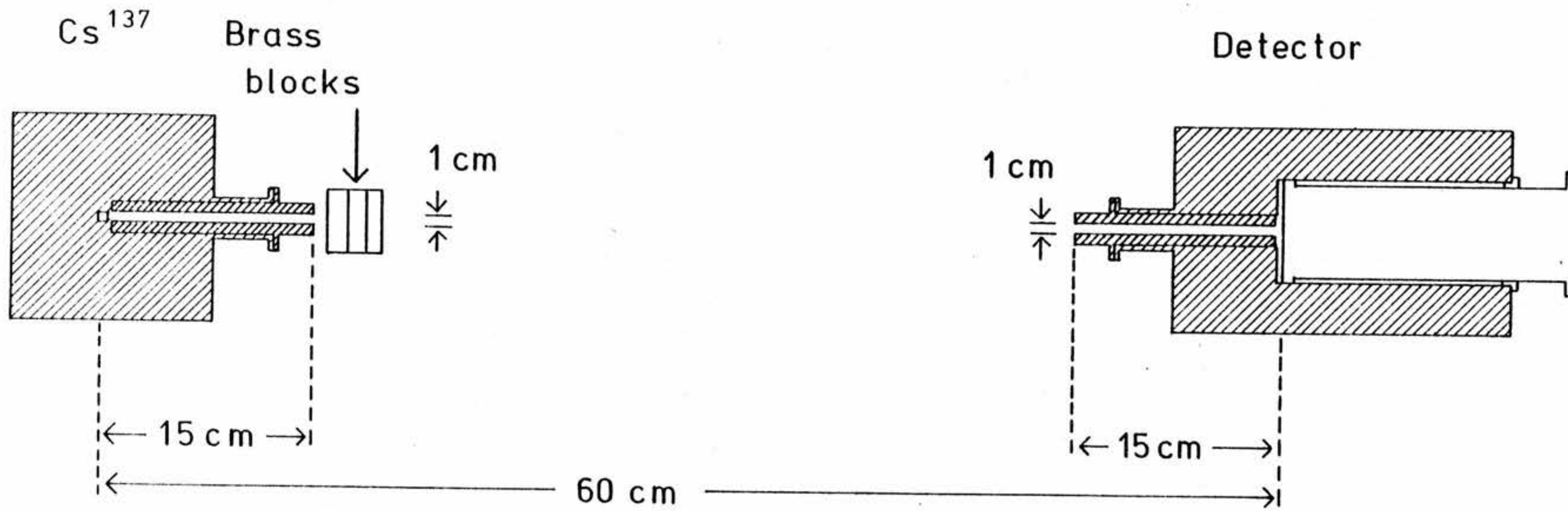
Block	Thickness
A	3.745 ± 0.001
B	3.741 ± 0.002
D	2.620 ± 0.002
E	1.120 ± 0.010

fig. 4·16 : Attenuator thickness

Blocks	Total Count	Counting Time (secs.)	Thickness (cm)	Count Rate (k counts/sec)
ABE	1 010 884	1000	8.612	1.011
AB	1 005 453	500	7.486	2.011
AD	1 205 289	300	6.365	4.018
AE	1 008 035	100	4.874	10.08
A	1 022 135	50	3.745	20.44
D	1 234 697	30	2.620	41.16
E	1 036 262	10	1.126	103.6
-	1 057 806	5	0.000	211.6

errors in counting time are negligible hence errors in count rate (1σ) are $< 0{\cdot}1\,\%$.

fig. 4·17 : Count rate vs. attenuator thickness



Outer casing and collimators for source and detector are lead.

fig. 4·18 : Linearity check, physical layout

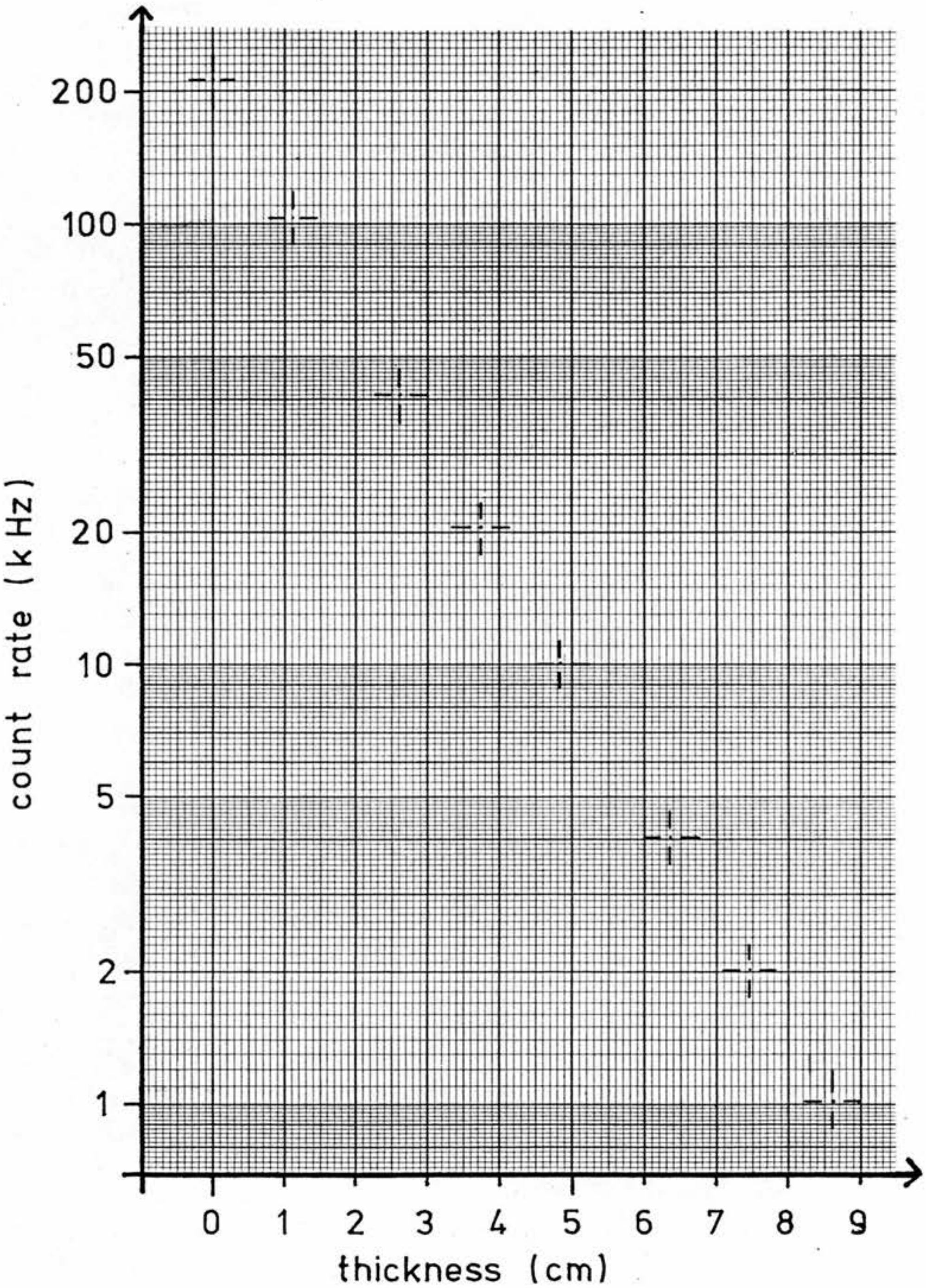


fig. 4·19 : Linearity check

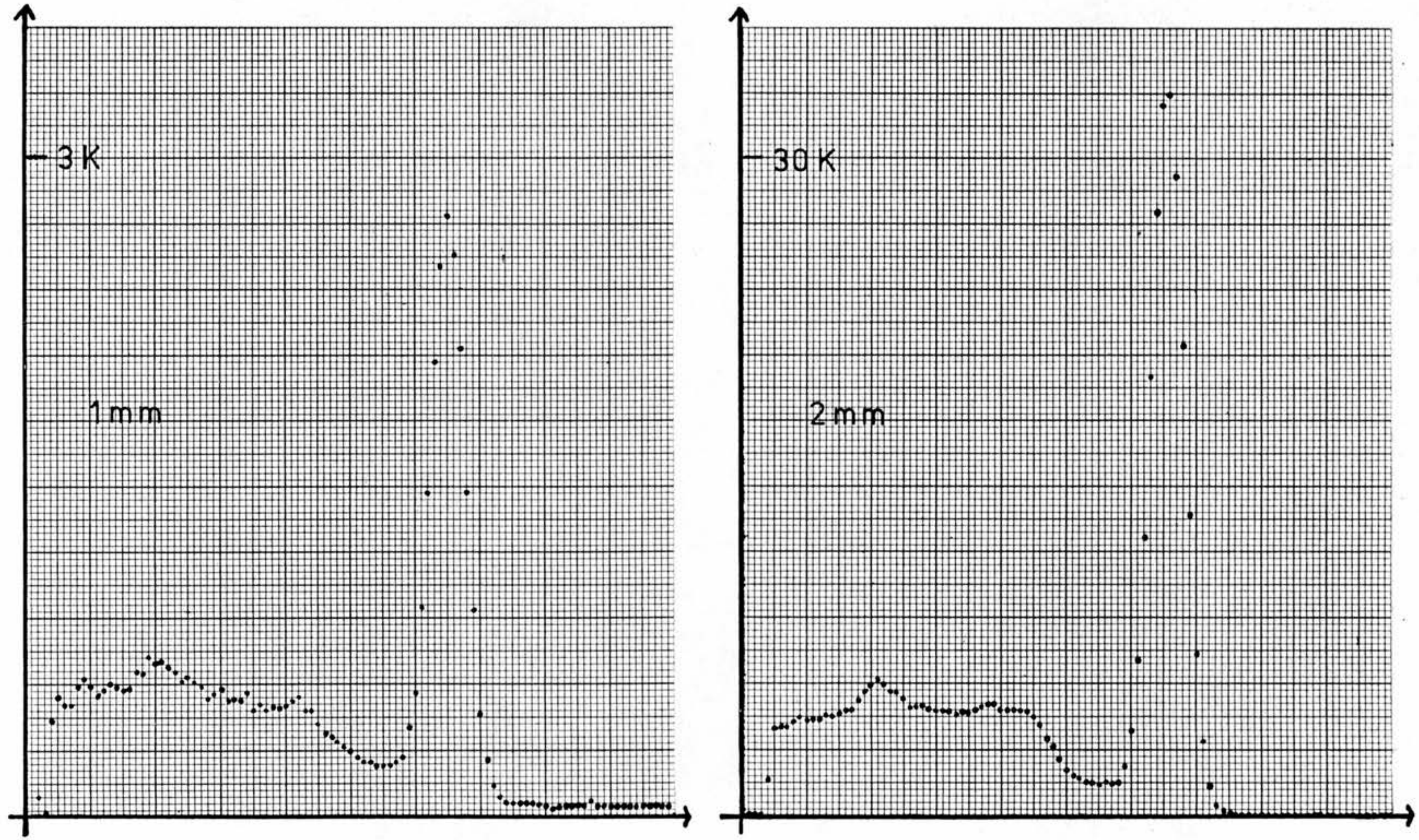


fig. 4·20(a) : Detector spectra, no pulse shaping

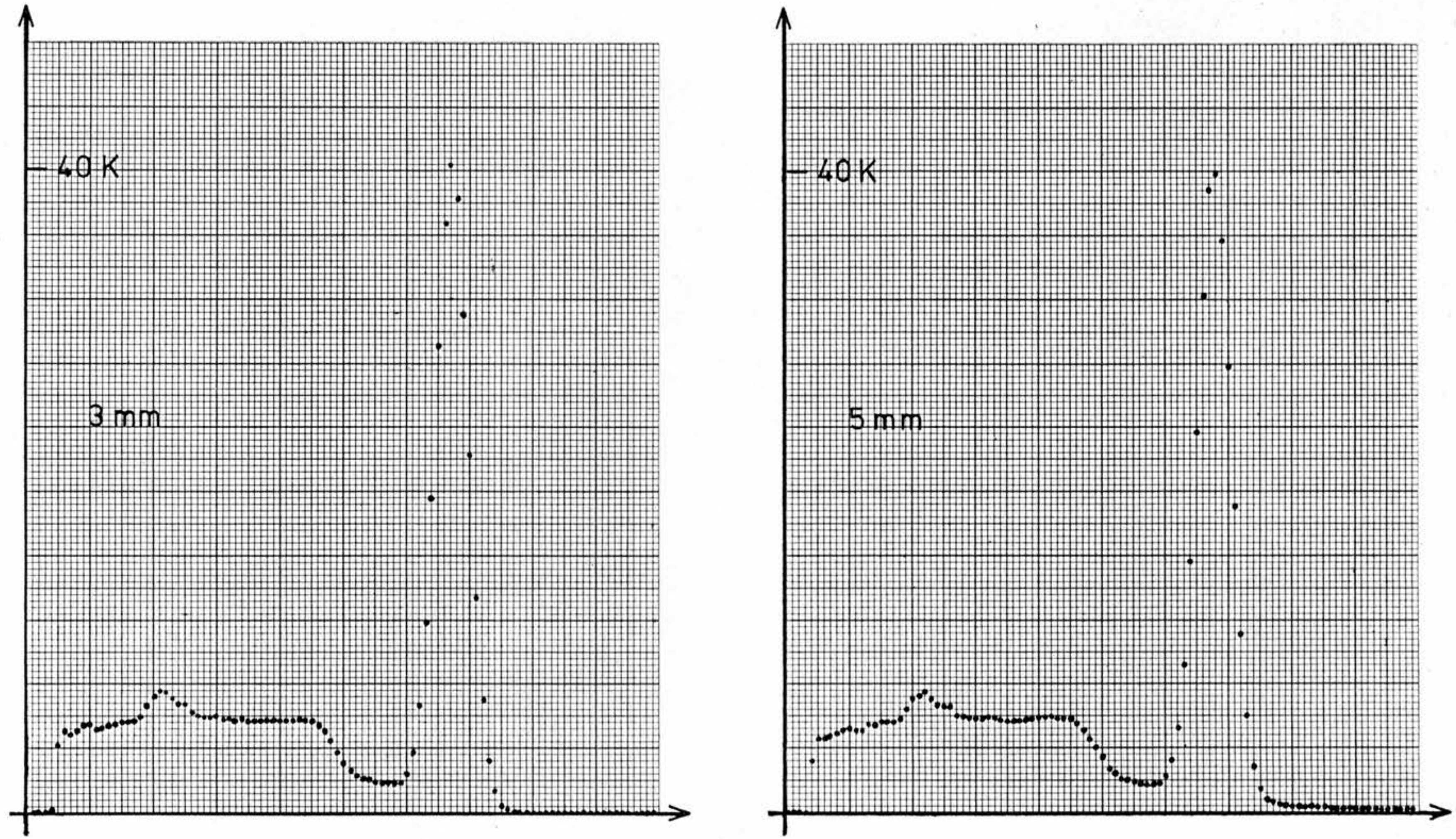


fig. 4·20(b) : Detector spectra, no pulse shaping

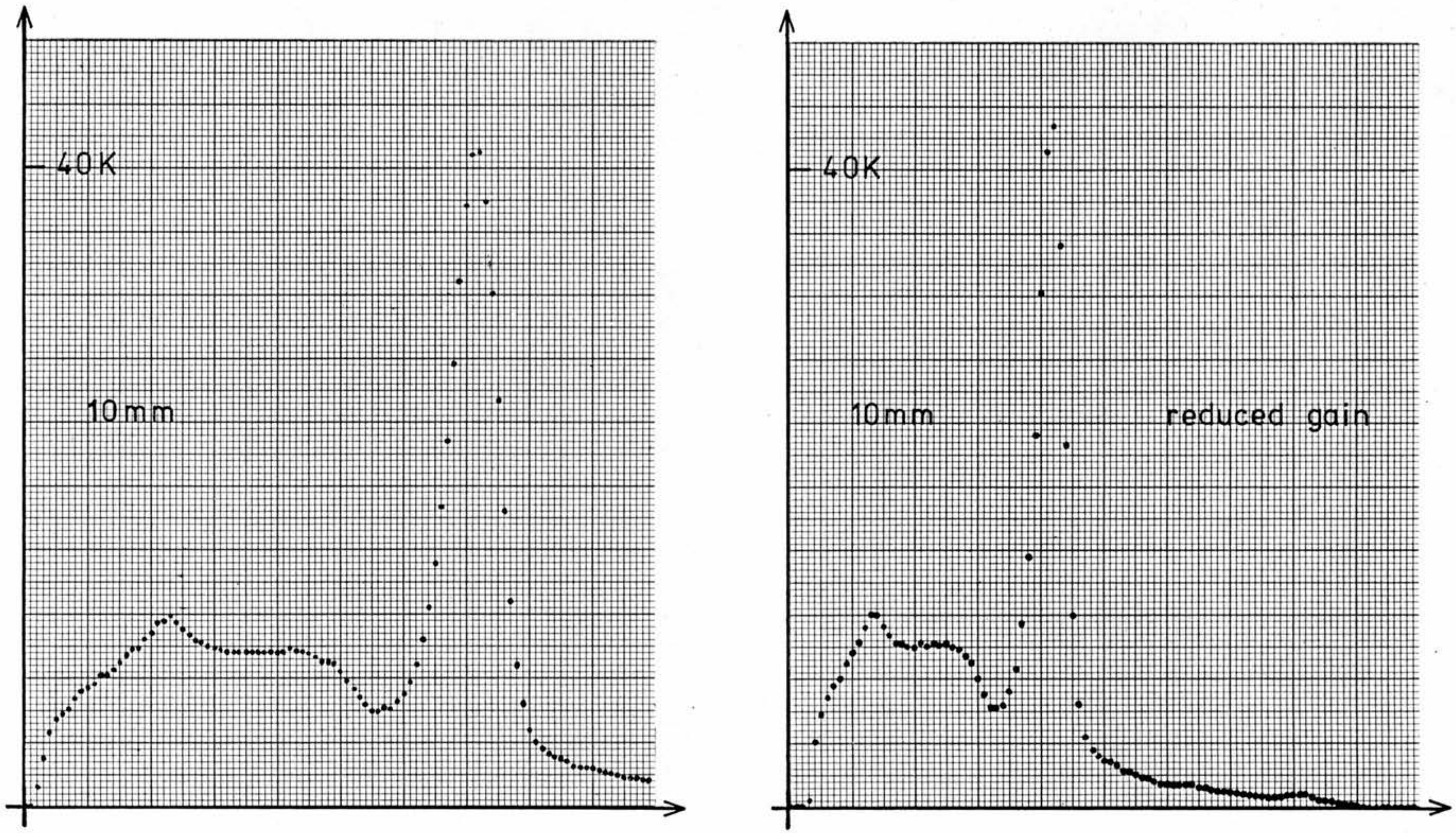


fig. 4·20(c) : Detector spectra, no pulse shaping

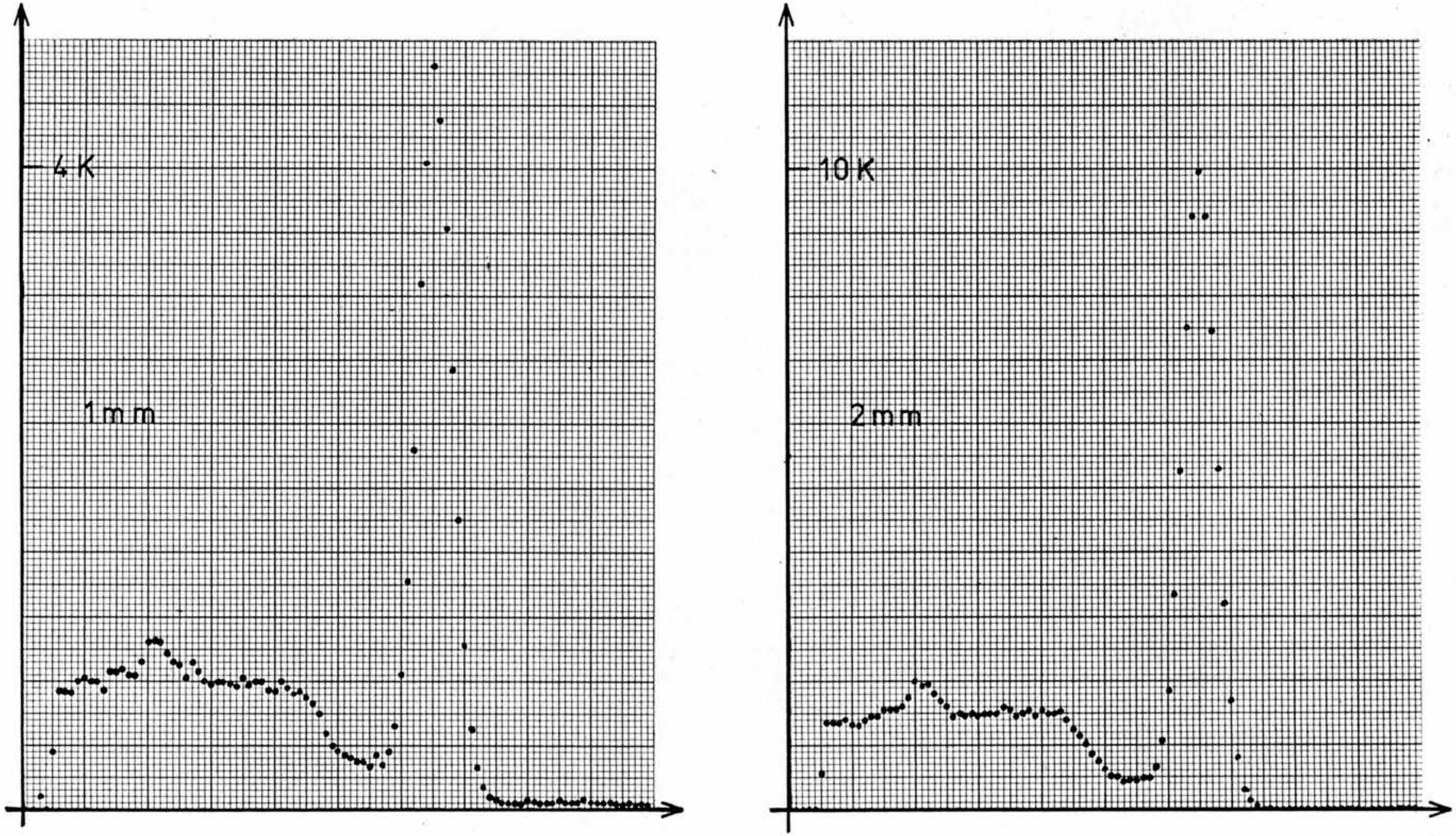


fig. 4·21 (a) : Detector spectra, with pulse shaping

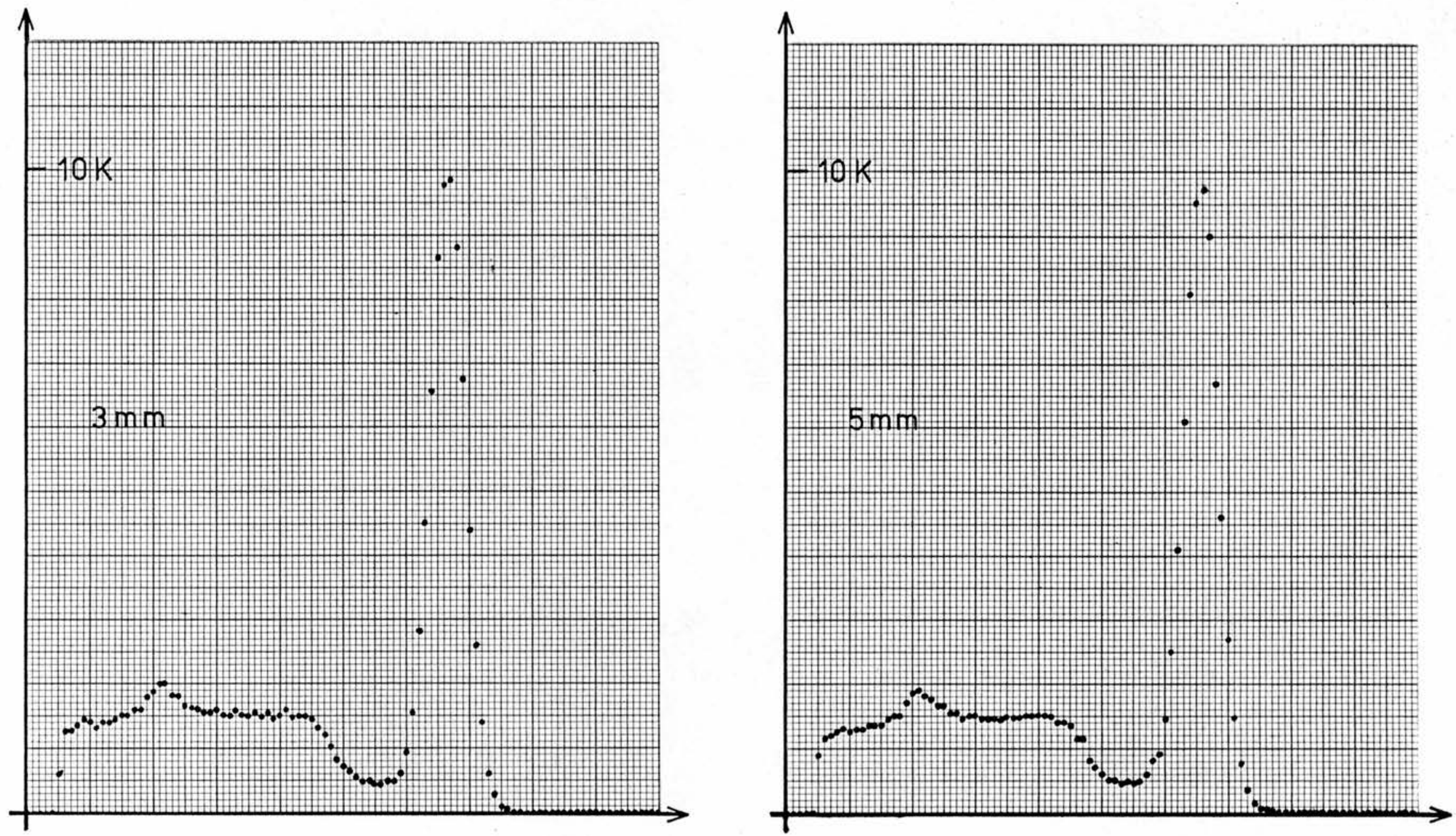


fig. 4·21(b) : Detector spectra, with pulse shaping

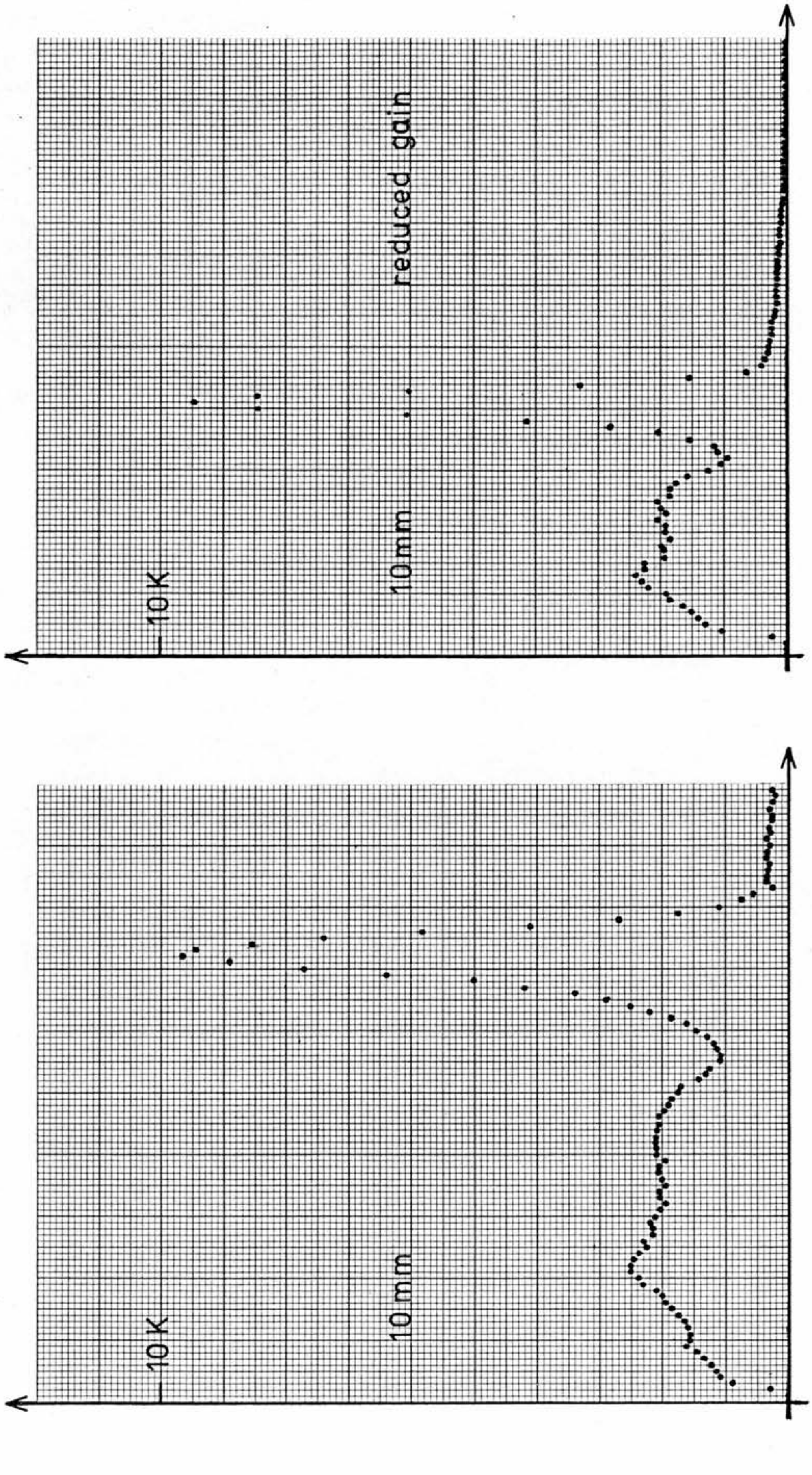


fig. 4·21 (c) : Detector spectra , with pulse shaping

The MECHANICAL CONSTRUCTION of the detector is illustrated in F·22 - F·24 The detector is contained in a brass cylindrical casing (roughly 3" in diameter and 12" long). From right to left can be seen, the NaI(Th) crystal, the μ-metal screen around the PM tube (both held at -ve HT potential) and the printed circuit board on which are mounted the dynode chain and the pulse shaping network. At the left are two screened compartments containing the head amplifier and HT filter (R" and C). The construction incorporates the usual spring loading to ensure good contact between crystal and PM tube. The p.c. board is double sided, having wiring on one side and earth plane and components on the other with the dynode chain and pulse shaping network kept as far apart as possible. Wires from the dynode chain to PM tube were about 1" long and did not cause problems, but screened cable was found to be necessary for all signal leads and some experiment required to establish the best internal earthing arrangements. Once the final construction had been achieved the detector performed reliably and the earlier tendency of the head amplifier to oscillate was no longer observed.

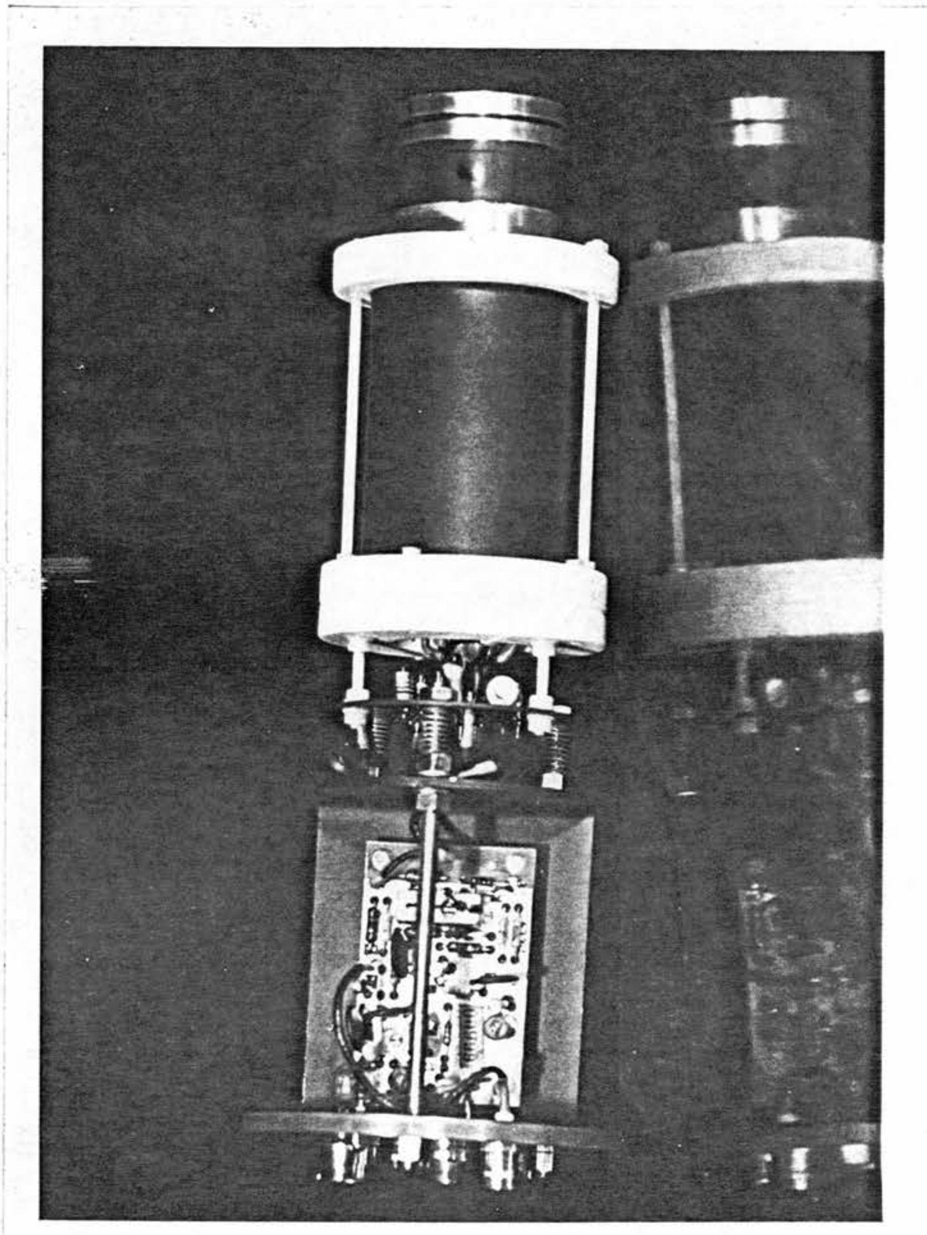


fig. 4·22 : The detector

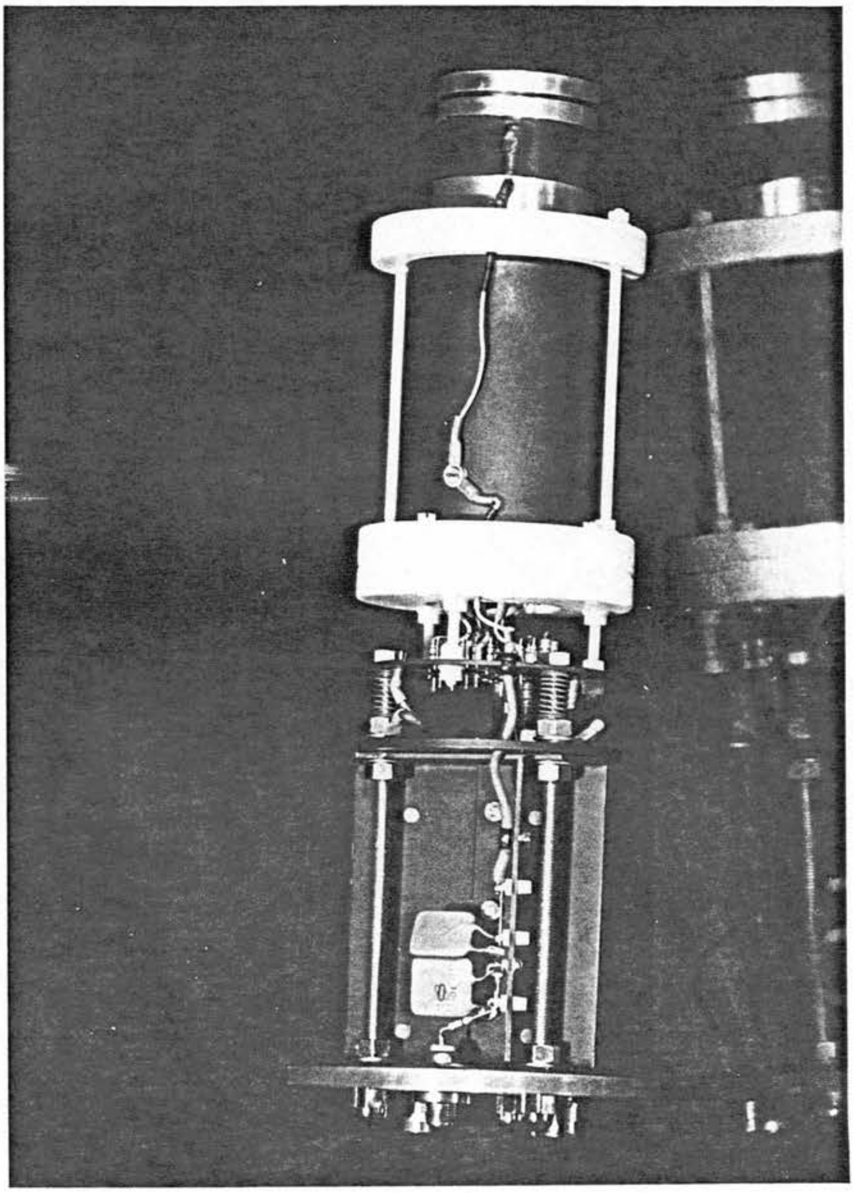


fig. 4·23 : The detector

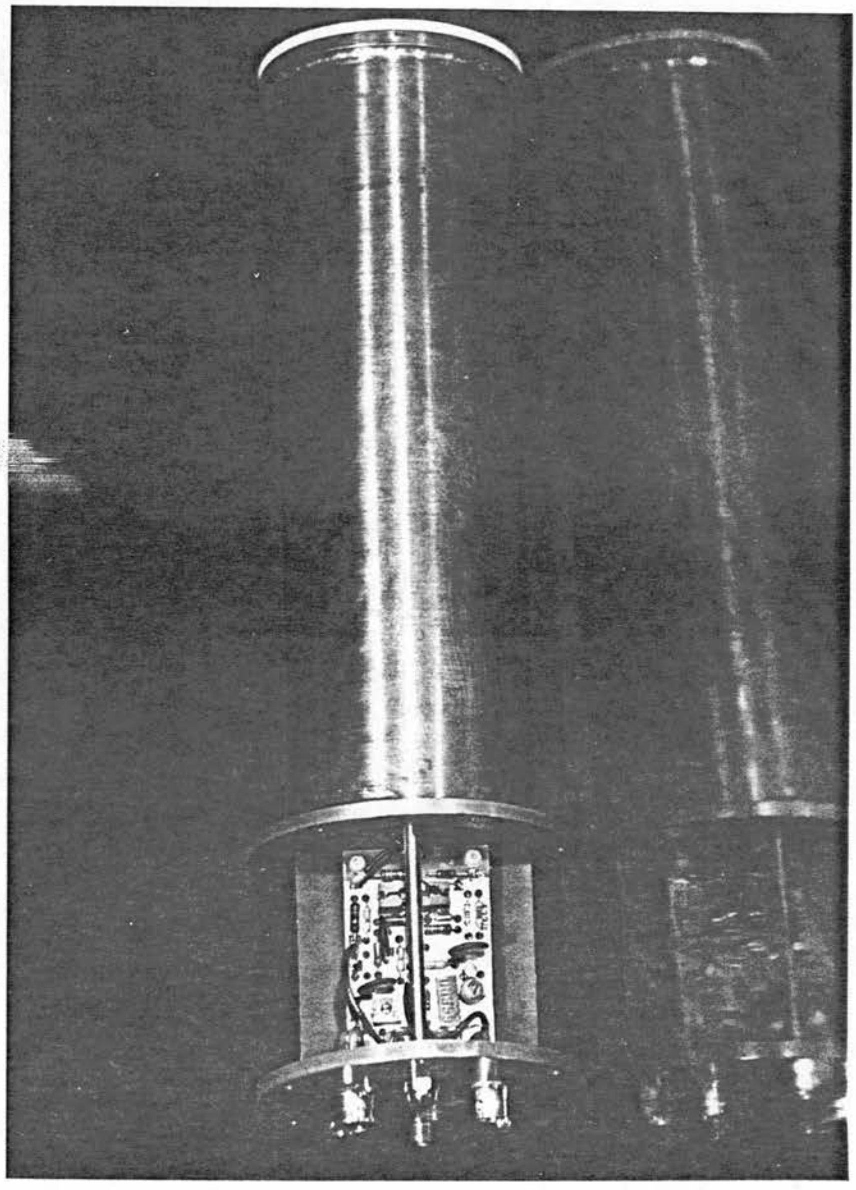


fig. 4·24 : The detector

Chapter 5 : Software

·1 Introduction

In this chapter a brief description of the software is given. Because of the quantity of software developed (about 15,000 lines of high level code) it is impossible to give a comprehensive account of it in the space available. For this reason only the following are given: the overall software requirements (§·2), an overview of the software design (§·3) and a more detailed software design for some of the main programs (§·4).

·2 Software requirements

There were six requirements placed on the software which was written to go with the scanner:-

·21 Prediction software was required to show what data should be collected (i.e. implementing the ideas of §3).

·22 PDP 12 software was required to drive the computer interface outlined in §4·35.

·23 Data processing software was required to implement the results of §2.

·24 It was required that the data calculated at any one of several predetermined points in the data processing software (§·23) could be output in one of two forms:-

- ·241 graphically: as either a grey scale picture or an isometric projection.
- ·242 numerically: as a straightforward data dump in decimal, octal or hexadecimal.

·25 At a number of points in the data processing decisions are made on how the processing should proceed next and these decisions can only be made at run time. It was therefore required that the data processing software (§·23) should contain code which would allow a complete traceback of the processing performed on any given data set.

·26 When working with limited software development resources there is a trade off between writing reliable code and writing efficient code. In a research project such as this it was felt that the overriding requirement was for reliable code.

·3 Software design - general

·31 Hardware considerations

Two computers were considered for use in the data processing: a PDP 12 (available within the department) and a large regional computer system (EMAS, which was remote from the department).

The PDP 12 had 8K of memory, used FOCAL (an interpretative language) and had only a teletype for hard copy output and no program libraries were available.

The regional system, EMAS, was a multi-access sytem (based on two I.C.L. 4-75 computers) offering 13M bytes of virtual memory to each user, various high level compilers, with line printer, graph plotter and matrix plotter outputs and supporting a range of program libraries.

It was felt that EMAS was better suited to the software requirements as outlined in §·2 and the majority of work was therefore done on this system. The PDP 12 was used only as a data logging system for the scanner (the data being placed on LINC tape and then transferred to EMAS, see F·1).

·32 Programming language

The EMAS system offers IMP, FORTRAN IV and ALGOL 60 computers. Of these three IMP (a locally developed language which is now available on newer ICL machines) offers particularly attractive facilities because it is not only

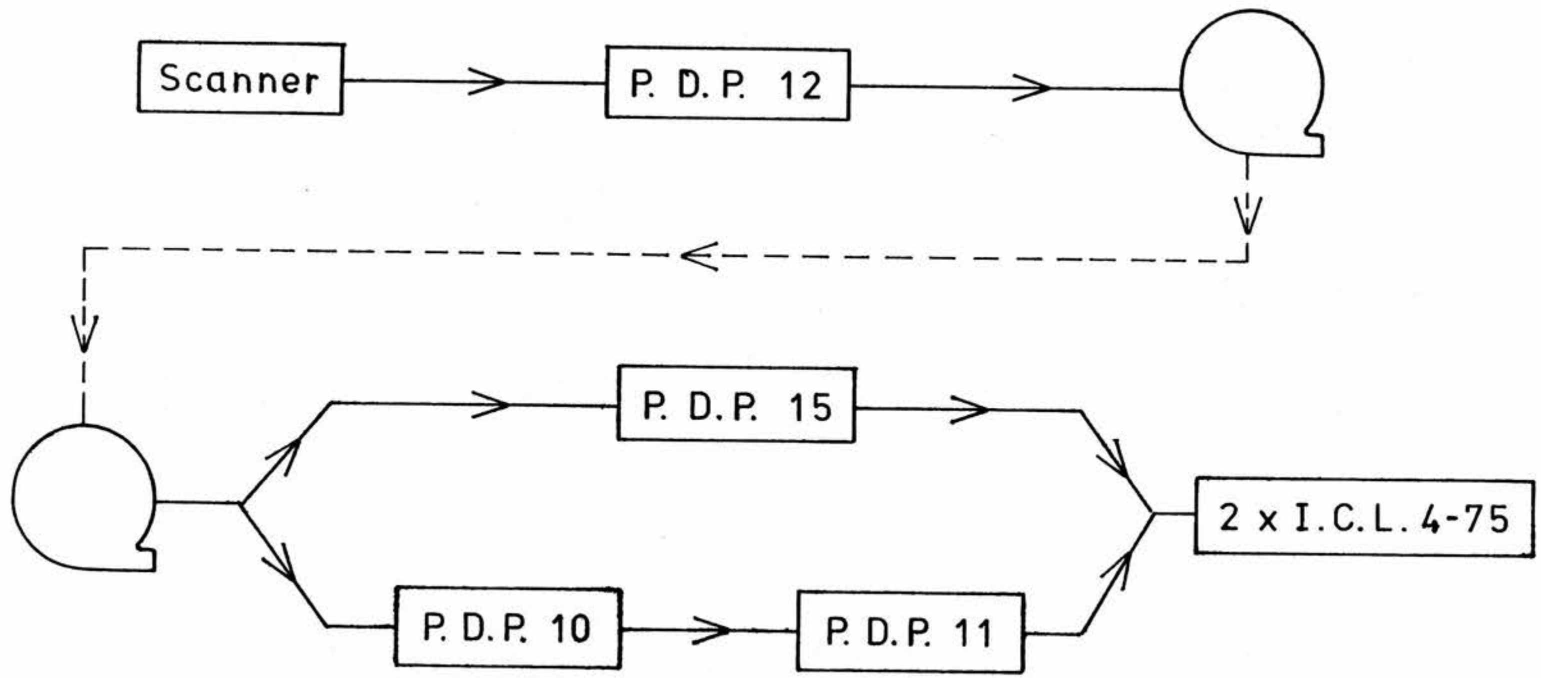


fig. 5-1 : Scanner - EMAS data path.

available to the user as a high level programming language but is also the language in which the EMAS operating system is written. The effect of this is that the user can access all the operating system commands from within a program (simply by declaring them as external routines) thereby making such things as file handling and I/O channel definitions particularly painless. In addition, IMP offers records, store mapping, easy bit pattern and character manipulation and recursive use of routines (see McLeod, 1978).

The advantages offered by these facilities were considered to outweigh the disadvantage of the language being non-transportable and IMP was therefore used for all the programming other than that required in §·22.

·33 Overview of software structure

·331 Main data processing software

In §2 it was seen that the process of passing from the measured data $\underline{m}(\phi_j)$ to an estimate $\hat{f}$ fell naturally into four parts. For the purposes of software development the last of these $(p \to f)$ is split into two parts $(p \to s * p$ and $s * p \to f)$ giving a total of five mappings.

The five inverse mappings together with the software implementing them are tabulated in the first two rows of F·2.

·332 Data and parameter files

At each of the five stages in the data processing new partially processed data is produced. For example, the program GQ uses the estimates of g as a basis for estimating q and this new data is placed in a file ESTQ (ESTimate of Q). In addition parameters showing how the program GQ has run (i.e. giving the traceback facility required in §·25) are placed in the file PGQ (Parameters from GQ). A similar nomenclature is applied to the other mappings and is summarised in the last two rows of F·2.

mapping	program	data file	parm. file
	REHASH		
	TRIM	ESTM	PTRIM
$\underline{m} \longrightarrow \underline{\hat{g}}$	MG	ESTG	PMG
$\underline{\hat{g}} \longrightarrow \underline{\hat{q}}$	GQ	ESTQ	PGQ
$\underline{\hat{q}} \longrightarrow \underline{\hat{p}}$	QP	ESTP	PQP
$\underline{\hat{p}} \longrightarrow \widehat{\underline{\underline{s}}\,\underline{p}}$	PSP	ESTSP	PPSP
$\widehat{\underline{\underline{s}}\,\underline{p}} \longrightarrow \underline{\hat{f}}$	SPF	ESTF	PSPF

fig. 5·2 : Main program and file names.

·333 Output facilities

Various output facilities are provided. These were developed partly as output routines intended for use in the final software system and partly for use as debugging aids during software checkout.

·3331 Data file output

Since the data files are large it is not always productive to look at a straightforward numerical dump of their contents (simply because of the quantity of numbers involved). For this reason several forms of output were developed.

1) Isometric projection of a surface in three dimensions (based on standard library routines).
2) Greyscale picture drawn on a Versatek matrix plotter (based on standard library routines).
3) Graph of a 1-D function (based on standard library routines).
4) Numerical data dump in octal, decimal or hexadecimal.

The format of the data files is sufficiently compatible for the same output routines to be applicable to all of them (except ESTM in some cases).

·3332 Parameter file output

These files each had their own different format as required by the details of the individual programs. Any of them may be dumped by use of a program LISTPARM which outputs a description of the parameter as well as its numerical value for each parameter in the relevant file.

·334 Batch mode facilities

For development purposes it was necessary to be able to run the software interactively, while for routine use it had to be run in batch mode. During the running of the software the user must make a number of decisions (e.g. in the running of PSP what action is to be taken if $\lambda_d > \lambda_{d\ max}$). When running in batch mode the user must predetermine what action is to be taken at all relevant points in the software. The program PRESET is used to do this by presenting a list of all the possibilities and requiring that the operator determine what action is to be taken in the different circumstances, these decisions are then recorded in a control file which is accessed when the data processing software is run in batch mode.

The program PRESET is also used to list the current state of the control parameters.

·335 Other software

In addition to the main data processing software and the output software discussed in §·331 - §·334 there are also various other major programs which are described below.

·3351 PREDICT: This program is used to predict how a section should be scanned in order to achieve a predetermined cut-off frequency in the reconstruction. It meets the requirement given in §·21 and is based on the ideas developed in §3.

·3352 SH1: This programe is used to drive the scanner interface outlined in §4·35, is run on the PDP 12 and written in PAL 12 assembler. It was written by Dr A.K. Boardman and is the only program not run on EMAS.

·3353 REHASH: As a result of the rather circuitous route used to transfer data from the PDP 12 to EMAS (see F·1), the bit pattern format of the data arriving on EMAS was very different from the standard EMAS format. This program is used to restore the bit patterns to a form intelligible to EMAS system.

·3354 TRIM: This program is used to trim off any excess sample points in the m data, (these may arise if the traverses are not all started and stopped at exactly the same points) and also to reverse the order of the sample points in every other traverse thereby ensuring that in all traverses the sample points corresponding to negative α (§1·2 E·10) occur earliest in the data file.

·3355 COMPARE: This program is used to calculate the $\mathcal{L}^1, \mathcal{L}^2$ and $\mathcal{L}^\infty$ norm of the difference between the original section used in the prediction program (§·3351) and the reconstructed section given by the data processing software.

·34 Reliability considerations

As noted in §·26, it was felt that the overriding requirement was for reliable software and a number of standard techniques were invoked in order to achieve this. These are listed below.

1. High level language. With the exception of the program SH1 all the software was written in IMP (see §·32).

2. Structured coding. The IMP language supports the IF-THEN-ELSE, DO-WHILE and DO-UNTIL logical controls. In addition multibranch selection between operations (switch vectors) are also available. As a result the GOTO type control is virtually unused.

3. Programming standards. The discussion of §·331 and §·332 will have suggested that fairly rigid programming standards were applied to the naming of major programs and files. This was also true of the names used for variables within programs. In general variables were assigned names as close as possible to those used in the mathematics developed in chapters 2 and 3 and much use of store mapping (equivalencing of variable names) was made in order to achieve this.

4. Modular programming was used extensively.

5. Hierarchical programming was not found to be very relevant, the software being better viewed as sequential (i.e. the five mappings of §·331 applied

in sequence). However, a top-down procedure was found useful in a few cases, notably LISTPARM and PRESET.

6. Debugging aids used were:-

   a) Line by line checking of the code by the programmer.

   b) Source language debugging (provided by the compiler)

   c) Use of optional snap shot dumps of all scaler variables when run time errors are encountered. These are provided partly by code implanted by the compiler and partly by failure routines written as part of the software and called under various fault conditions.

   d) Using the software with specially generated test data.

·4 Individual program designs

·41 Introduction

In §·3 an overview of the main software has been given. In total there were

20 main routines

41 general utility routines

10 graphics routines

11 file manipulation routines

and these, together with sundry other pieces of software, add up to about 15,000 lines of code. In the space available there is no possibility of giving even detailed flow diagrams for the main routines let alone software listings and all that is attempted in the following sections is to summarise the formulae implemented by the main data processing routines.

The main software routines (and the relevant section numbers where appropriate) are tabulated below.

(1) Setting up scanner

phantom programs - no details given

PREDICT - §3·5·,(see also chapter 3 and its other appendices).

(2) Scanner operation

SH1 - §·3352

(3) Data preprocessing

REHASH - §·3353

TRIM - §·3354

(4) Data processing

MG - §·42 (see also §2·2)

GQ - §·43 (see also §2·3)

QP - §·44 (see also §2·4)

PSP - §·45 (see also §2·5)

SPF - §·46 (see also §2·5)

(5) Data assessment

COMPARE - §·47

Graphics -§·3331

Dumping -§·3331

·42 Program MG

This program must calculate ℓ and then use this value to calculate $\hat{\underline{g}}$. In §2·222 it was seen that one calculates an estimate $\hat{\ell}$ rather than the exact value ℓ, and that if the use of $\hat{\ell}$ in place of ℓ was to introduce an error of less than θ in the estimate $\hat{\underline{g}}$ then $\hat{\ell}$ must be calculated as

$$\hat{\ell} = \frac{1}{N_1} \sum_{j=1}^{N_1} m_j \quad \text{where} \quad N_1 > 36/\ell\theta^2$$

where the m_j are counts measured with no absorbing medium between source and detector.

In practice data for calculating $\hat{\ell}$ was collected at the beginning of a scan by performing several traverses before placing the section on the scanner. The only question is how many such blank lines should be collected;

and this is easily determined during the prediction process. Once the operator has decided on a value for $\dot{\theta}$ then N_1 can be chosen as $36/\ell\theta^2$. (ℓ being already determined during the prediction process). Bearing in mind that each traverse has N cells of length Δr (both already known from the prediction process) and that this gives a traverse length of L, then

$$\text{no. of blank lines} = \frac{N_1}{N} = \frac{36\Delta r}{L\ell\theta^2}$$

and this is rounded up to give an integer value. In practice θ was always assigned the same value as the aliasing error θ_1.

Thus, given a number of blank lines preceding the data, the program MG calculates

$$\hat{\ell} = \frac{1}{N_1}\sum_{j=1}^{N_1} m_j$$

where N_1 = no. of blank lines × N

(i.e. using all the data from the blank lines) and then calculates $\underline{\hat{g}}(\phi_j)$ as $\underline{\hat{g}}(\phi_j) = \hat{\ell}^{-1}\underline{m}(\phi_j)$.

Finally, in preparation for the use of F.F.T. algorithms in later programs, the vector $\underline{\hat{g}}(\phi_j)$ is padded out into a vector of N_2 components. Define

$$N_2 = \min\{2^n : 2^n \geq N \text{ and } n \in \mathbb{N}\}$$

then

$$(\underline{\hat{g}}_{N_2}(\phi_j))_k = \begin{cases} 1 & 0 \leq k \leq \frac{N_2-N}{2} - 1 \\ (\underline{\hat{g}}(\phi_j))_{\left(k - \frac{N_2-N}{2}\right)} & \frac{N_2-N}{2} \leq k \leq \frac{N_2+N}{2} - 1 \\ 1 & \frac{N_2+N}{2} \leq k \leq N_2 - 1 \end{cases}$$

where $\underline{\hat{g}}_{N_2}$ denotes the padded out version of $\underline{\hat{g}}$. In other words the N-vector is increased to an N_2-vector by padding it out with $N_2 - N$ components of value 1, half of them being placed at the beginning of the vector and half at the end. The value 1 is chosen since $g(r) = 1 \quad |r| \geq a + D/2$ where a is the collimater radius and D is the section diameter.

·43 Program GQ

This program functions in one of two ways performing either the complete deconvolution as discussed in §2·3 or just creating the necessary files so that the program QP thinks that deconvolution has been performed (the reasons for wishing to do this may be found in §3·1). The process of deconvolution may be considered as finding a matrix $\underline{\underline{\ell}}$ and then using it to estimate $\hat{\underline{q}}$ as

$$\hat{\underline{q}} = \underline{\underline{\ell}}\,\hat{\underline{g}}\ .$$

Under these circumstances

$$\underline{\underline{V}}(\hat{\underline{q}}) = \underline{\underline{\ell}}\ \underline{\underline{V}}\ (\hat{\underline{g}})\underline{\underline{\ell}}'$$

and $\underline{\underline{V}}(\hat{\underline{q}})$ is therefore estimated as

$$\hat{\underline{\underline{V}}}(\hat{\underline{q}}) = \underline{\underline{\ell}}\ \hat{\underline{\underline{V}}}(\hat{\underline{g}})\underline{\underline{\ell}}'$$

$$= \ell^{-2}\ \underline{\underline{\ell}}\ \mathrm{diag}(\underline{m})\underline{\underline{\ell}}' \qquad \text{(see §2·23)} \quad .$$

When GQ is used in its second mode (so that the correct files etc. are available to QP) then, in effect, one has simply put $\underline{\underline{\ell}}$ equal to the identity matrix so that

$$\underline{\underline{V}}\ (\hat{\underline{q}}) = \underline{\underline{V}}\ (\hat{\underline{g}})$$

and $\underline{\underline{V}}(\hat{\underline{q}})$ is therefore estimated as

$$\hat{\underline{\underline{V}}}(\hat{\underline{q}}) = \ell^{-2}\mathrm{diag}(\underline{m}).$$

These facts will be required later.

When discussing various statistical details of the program, it must be remembered that $\hat{\underline{g}}$ has been padded out to an N_2-vector but that the additional components are constants and therefore not subject to statistical error. For this reason summations over the components of $\hat{\underline{g}}$ will appear where only those components corresponding to the original N-vector are included.

In this section the quantity $\hat{\underline{g}}$ will always denote the N_2-vector.

·431 Full run of GQ

The information available at the start of the program is:-

$\hat{\underline{g}}$ from the program MG

and

$$H(R) = 2\,\frac{\sin(\pi\Delta rR)}{\pi\Delta rR}\;\frac{J_1(2\pi aR)}{2\pi aR} \qquad \text{from §2·36.}$$

The calculation proceeds as follows:-

1) calculate $\widehat{e^2} = \ell^{-1} \displaystyle\sum_{k=\frac{N_2-N}{2}}^{\frac{N_2+N}{2}-1} \hat{g}_k$

$$= \ell^{-1}\left\{\sum_{k=0}^{N_2-1} \hat{g}_k - (N_2 - N)\right\}$$

2) calculate $\hat{\sigma}^2(e^2) = 2\ell^{-2} \displaystyle\sum_{k=\frac{N_2-N}{2}}^{\frac{N_2+N}{2}-1} \hat{g}_k^2$

$$= 2\ell^{-2}\left\{\sum_{k=0}^{N_2-1} \hat{g}_k^2 - (N_2 - N)\right\}$$

so that $2\hat{\sigma}(e^2) = \dfrac{2\sqrt{2}}{\ell}\left\{\displaystyle\sum_{k=0}^{N_2-1} \hat{g}_k^2 - (N_2 - N)\right\}^{1/2}$

3) set

$$H_k^* = \begin{cases} \dfrac{1}{\sqrt{N_2}}\,H(k\Delta R) & 0 \le k \le N_2/2 \\ H^*_{N_2-k} & N_2/2 < k \le N_2 - 1 \end{cases}$$

4) set $\underline{c} = (-2,1,0,\ldots,0,1)'$ and calculate $\underline{C} = \underline{\underline{W}}\,\underline{c}$

5) calculate $\hat{\underline{G}} = \underline{\underline{W}}\,\hat{\underline{g}}$

6) find a value γ such that:-

$$\left|\widehat{e^2} - \sum_{k=0}^{N_2-1}\left(\frac{\gamma\overline{C}_k C_k}{\overline{H_k^*}H_k^* + \gamma\overline{C}_k C_k}\right)^2 \overline{\hat{G}}_k\hat{G}_k\right| \le 2\hat{\sigma}(e^2)$$

7) calculate $\underline{\underline{L}} = N_2^{-1}(\underline{\underline{H}}^{*T}\underline{\underline{H}}^* + \gamma\,\underline{\underline{C}}^T\underline{\underline{C}})^{-1}\underline{\underline{H}}^{*T}$

8) calculate $\hat{\underline{Q}} = \sqrt{N_2}\,\underline{\underline{L}}\,\hat{\underline{G}}$

9) calculate $\hat{\underline{q}} = \underline{\underline{W}}^{-1}\hat{\underline{Q}}$

and $\underline{\underline{\ell}} = \underline{\underline{W}}^{-1}\underline{\underline{L}}$

The main data output is $\hat{\underline{q}}$, $\underline{\underline{\ell}}$ is also saved for later use in connection with e_d^2 in the inversion of $p = \mathcal{R} f$.

·432 Simulated run of GQ

When a simulated run of GQ is performed (i.e. no deconvolution) then no actual calculation is done and the routine performs only computer housekeeping.

·44 Program QP

This program inverts the mapping $p \to q$ by using the relation $p = -\ln q$. The program allows for non-positive values of q by assigning a minimum permitted value for q of 0·01 which corresponds to attentuation of the beam by about 54 cms. of water (clearly the particular number must be chosen according to the scanner design). Any values of $\hat{q}$ below this minimum are logged by the software and may be subsequently examined by the operators.

Input data required $\hat{\underline{q}}$ from the program GQ

Calculation performed $\hat{p}_k(\phi_j) = -\ln\left(\max\{\hat{q}_k(\phi_j), 0{\cdot}01\}\right)$

Main output $\hat{\underline{p}}$ together with log of small $\hat{\underline{q}}$ values.

·45 Program PSP

This program estimates the quantity $s * p$ as outlined in §2·4. The details of this calculation are affected by whether GQ has been run so that deconvolution was actually performed or just simulated.

In §2·42 E·3 it was stated that $\underline{\underline{V}}(\hat{\underline{p}})$ would be estimated by

$$\hat{\underline{\underline{V}}}(\hat{\underline{p}}) = \mathrm{diag}(\hat{\underline{q}})^{-1}\hat{\underline{\underline{V}}}(\underline{q})\,\mathrm{diag}(\hat{\underline{q}})^{-1}$$

and combining this with the equations of §·44 gives the following.

For a full run of GQ

$$\hat{\underline{\underline{V}}}(\hat{\underline{p}}) = \ell^{-2}\mathrm{diag}(\hat{\underline{q}})^{-1}\underline{\underline{\ell}}\,\mathrm{diag}(\underline{m})\,\underline{\ell}'\ \mathrm{diag}(\hat{\underline{q}})^{-1}$$

(see however the closing paragraph of §2·42), and for a simulated run of GQ

$$\hat{\underline{\underline{V}}}(\hat{\underline{p}}) = \mathrm{diag}(\ell^{-1}\underline{m})^{-1}\ \ell^{-2}\mathrm{diag}(\underline{m})\,\mathrm{diag}(\ell^{-1}\underline{m})^{-1}$$

$$= \mathrm{diag}(\underline{m})^{-1}$$

Given these facts the calculation performed by the program may be summarised as follows.

Main information required at start of program:-

following full run of GQ $\hat{\underline{p}}$, ℓ, $\hat{\underline{q}}$, $\underline{\underline{\ell}}$, $\underline{m}$,

following simulated run of GQ $\hat{\underline{p}}$, ℓ, $\underline{m}$,

and in either case formulae for $\lambda_{d\ min}(\theta_2)$ and $\lambda_{d\ max}(m,N_3)$ are also needed.

Formulae used are:-

(a) $$e_d^2 = \sum_{j=1}^{M} \mathrm{tr}\ \underline{\underline{V}}(\hat{\underline{p}}(\phi_j))$$

(b) $$\sum_{j=1}^{M} [\,\underline{\underline{\Lambda}}^5(\underline{\underline{\Lambda}}^5 + \lambda_d\underline{\underline{I}})^{-1}\hat{\underline{P}}(\phi_j)]^T[\,\underline{\underline{\Lambda}}^5(\underline{\underline{\Lambda}}^5 + \lambda_d\underline{\underline{I}})^{-1}\hat{\underline{P}}(\phi_j)\,] = e_d^2$$

(c) $$N_3 \geq N + \left(\frac{\lambda_{d\ min}}{\lambda_{d\ N_2}}\right)^{1/5} N_2$$

(d) and the sampled version of $s * p$ is given by

$$\frac{1}{N_3\Delta r}\ \underline{\underline{W}}^{-1}\lambda_d\ \underline{\underline{\Lambda}}(\underline{\underline{\Lambda}}^5 + \lambda_d\underline{\underline{I}})^{-1}\hat{\underline{P}}(\phi_j)$$

The calculation proceeds as follows.

1) put $N_3 = N_2$

2) estimate e_d^2 by $\widehat{e_d^2} = \sum_{j=1}^{M} \mathrm{tr}\ \hat{\underline{\underline{V}}}(\underline{p}(\phi_j))$

For a simulated run of GQ this gives

$$\hat{e}_d^2 = \sum_{j=1}^{M} \sum_{k=0}^{N-1} m_k(\phi_j)^{-1}$$

by substituting for $\hat{\underline{\underline{V}}}(\hat{\underline{p}})$ from above. For a full run of GQ one has

$$\hat{e}_d^2 = \ell^{-2} \sum_{j=1}^{M} \sum_{n=0}^{N_2-1} \sum_{k=0}^{N-1} (\underline{\ell}(\phi))^2_{n - \frac{N_2-N}{2} - k|N_2} \quad (\underline{m}(\phi_j))_k \,/\, (\underline{q}(\phi_j))_n^2$$

3) calculate $\hat{\underline{P}} = \hat{\underline{\underline{W}}}\,\hat{\underline{p}}$

4) calculate $\lambda_{d\,N_2}$ from (b)

5) calculate $N_3' = N + (\lambda_{d\,min}/\lambda_{d\,N_2})^{1/5} N_2$

6) calculate $N_3 = \min\{2^n : 2^n \geq N_3' \text{ and } n \in \mathbb{N}\}$

7) calculate λ_d by α) pad out $\hat{\underline{p}}$ with zeros to give an N_3 - vector

β) calculate $\hat{\underline{P}} = \underline{\underline{W}}\,\hat{\underline{p}}$

γ) use (b) to calculate λ_d

8) check $\lambda_d < \lambda_{d\,max}(m, N_3)$: operator intervention required if otherwise.

9) check $\lambda_d > \lambda_{d\,min}(\theta_2)$: if not double N_3 and goto (7) otherwise goto (10).

10) calculate $s * p$ from (d).

The main output is the sampled version of $s * p$.

·46 Program SPF

This program implements the back projection process. From §2·531

$$\hat{f}(r,\phi) = \Delta\phi \sum_{\ell=1}^{M} \underline{v}(d_\ell)'\,\underline{t}(\phi_\ell)$$

where

$$d_\ell = \frac{r}{\Delta r}\cos(\phi - \phi_\ell) + \frac{N_3-1}{2}$$

and $\underline{t}$ is the sampled version of $s * p$.

If $\hat{f}$ is to be evaluated on an $N_4 \times N_4$ matrix symmetrically placed about the origin and with square cells of size Δr_2 then by reasoning identical to that used to arrive at §3·4311 E·21 (but applied to the reconstruction matrix instead of the phantom matrix) the j,kth. cell has Cartesian coordinates

$$\left(\left\{j - \frac{N_4 + 1}{2}\right\} \Delta r_2, \left\{k - \frac{N_4 + 1}{2}\right\} \Delta r_2\right)$$

and hence has polar coordinates

$$r(j,k) = \Delta r_2 \left[\left\{j - \frac{N_4 + 1}{2}\right\}^2 + \left\{k - \frac{N_4 + 1}{2}\right\}^2\right]^{\frac{1}{2}}$$

$$\phi(j,k) = \tan^{-1}\left[\left\{k - \frac{N_4 + 1}{2}\right\} \Big/ \left\{j - \frac{N_4 + 1}{2}\right\}\right].$$

Combining these equations $\hat{f}_{jk}$ can be estimated as

$$\hat{f}_{jk} = \hat{f}(r(j,k), \phi(j,k))$$

with $\hat{f}$ given by the above formulae.

The question of computation efficiency needs to be borne in mind when evaluating $\hat{f}_{jk}$. The obvious method is to choose j and k and then evaluate the formula but a little throught shows that this is most inefficient as it involves a large number of unnecessary evaluations of trigonometric functions. A better method is to concentrate on one projection (i.e. fix ℓ) and calculate its contribution to every cell of $\hat{f}_{jk}$ then move to the next projection and repeat the same process. This is the approach used by SPF.

Further details of the program are not given.

·47 Program COMPARE

This program is used to give measures (closely related to the $\mathcal{L}$, $\mathcal{L}^2$, and $\mathcal{L}^\infty$ norms) of the difference between the original phantom f_{jk} and the estimate $\hat{f}_{jk}$ calculated by the data processing software.

These measures are normalised to be independent of the cell size and number of cells (which are required to be the same for f and $\hat{f}$). The differences are output in both relative and absolute form, the relative difference being the difference expressed as a fraction of the original f_{jk}.

The formulae used are:-

$$L_{1\ abs} = N_4^{-2} \sum_{j,k=1}^{N_4} |f_{jk} - \hat{f}_{jk}|$$

$$L_{1\ rel} = L_{1\ abs} \Big/ \{N_4^{-2} \sum_{jk} f_{jk}\}$$

$$L_{2\ abs} = \{N_4^{-2} \sum_{jk} (f_{jk} - \hat{f}_{jk})^2\}^{\frac{1}{2}}$$

$$L_{2\ rel} = L_{2\ abs} \Big/ \{N_4^{-2} \sum_{jk} f_{jk}\}$$

$$L_{\infty\ abs} = \sup_{jk} |f_{jk} - \hat{f}_{jk}|$$

$$L_{\infty\ rel} = L_{\infty\ abs} \Big/ \{N_4^{-2} \sum_{jk} f_{jk}\}$$

Chapter 6: Computational Results

·1 Introduction

In this chapter various examples and results are given. These are the product of the work in chapters 2 - 5 inclusive.

The result of using the scanner and software is illustrated in §·2 which shows the outputs of various stages in the data processing from the starting point (the m data) to the final reconstruction (the f data). This is followed in §·3 by a discussion of two shortcomings of the scanner and its associated hardware.

The main aim of the chapter (contained in §·4) is to give some assessment of whether the mathematics of chapters 2 and 3 (or, more correctedly, the software to which it gave rise) does in fact function as intended and to this end four test objects were selected and used for prediction, scanning and reconstruction. These objects are introduced in §·41 and the results obtained from them are given in §·42 - §·48.

As a basic "figure of merit" assessment of a reconstruction, the $L_{2\ rel}$ measurement of the difference between a phantom and its reconstruction is used (see §5·47). Since this has the disadvantage of giving no idea what the reconstruction looks like a number of illustrations are also to be found. These are printed on a Versatek printer/plotter with software designed to give greyscale pictures with 16 greylevels (0 = white, 15 = black).

The chapter closes with the brief discussion of §·5.

·2 Example of reconstruction process

This section is devoted to describing and illustrating the results of using the reconstruction software outlined in chapter 5. There are two basic outputs from the software; the data files containing the partially or wholly processed data and the parameter files which provide the operator with a record of how the data was processed as well as various parameters calculated during the processing. In this section the data files are presented as greyscale plots (see §•1 and §5•3331) and the parameter files are listed using LISTPARM (§5•3332). (See F•1 and pp.185 - 197.)

The files illustrated are those named in F5•2 together with PPSPHIST (the parameter file that provides the complete trace back facility (see §5•25) for the program PSP), PSPCP and SPFCP (the files containing the Control Parameters for the batch mode running of PSP and PSF respectively).

The example is based on a scan of a tin of paraffin wax, however it should be clearly understood that details of the scan were determined by such requirements as fitting the illustrations on the page, etc., and the pictures and data should not therefore be taken as representative of a properly scanned object.

The following notes may halp the reader to understand how this example fits in with the discussion of chapters 2 - 5.

PTRIM (see §5•3354) - the parameters listed here are those which define how the scanner was used. The "stepping motor pulse frequency" is the setting of the pulse generator (§4•32) and hence is a measure of speed, the "gate pulse scale factor" is the 2^n scaling factor used to define the end of cell positions (§4•32) and hence the cell size, and the remaining parameters are self explanatory.

ESTM - in this plot, one line of the data across the greyscale plot represents one projection of the object. The eleven lines of data being two blanklines (i.e. no object between source and detector) used to calculate ℓ and nine projections (from different angles) of the tin of wax.

PMG - the "ℓ parameters" describe various details of the calculation of ℓ.

ESTG - note the dissappearance of the blanklines and the increase in the number of cells/line from $N = 52$ in the m data to $N_2 = 64$ in the g data in preparation for the deconvolution.

PGQ - if the full run of GQ is selected (as in this case) then each line (projection) of data is individually deconvolved. Thus each line gives rise to its own values of γ and $\omega_{c\ col}$ (both continuous and discrete versions, see §3·3·). These values are summarised statistically and given individually.

ESTQ - because of the poor quality of the data, in this particular case, deconvolution has made little difference.

PQP - the "small value parameters" refer to the possibility of q being negative, a situation which can arise as a result of ringing at discontinuities after deconvolution. Had any negative or small positive values been found, the software would have given details of their location (see §5·44).

PPSP - this is largely self explanatory when used in conjunction with §2·54 and §5·45.

ESTSP - note the increase from N_2 to N_3 data points.

PPSPHIST - it is instructive to consider this trace back of the program in order to appreciate what problems have been met by the software and how these have been dealt with. The program has started by putting $N_3 = N_2$ and has calculated a value for λ_d. To obtain some idea of what values are involved but with the minimum of computational effort only two projections have been used. This yields a rough estimate of $\lambda_d = 2{\cdot}831 \times 10^7$. The program now moves on to consider the constraint $\lambda_d \geq \lambda_{d\ min}$ (§2·5431) in combination with the requirement that N_3 must be sufficiently large for the convolution $s * p$ to be evaluated by circular (as opposed to linear) convolution (see §2·545 and in particular §2·545 E·4). These requirements are satisfied. The program now considers the constraint $\lambda_d \leq \lambda_{d\ max}$, and with $\lambda_{d\ max} = 2{\cdot}294 \times 10^3$ this constraint is violated. Inspection of the parameter file PSPCP shows that, in this situation, PSP is required to define λ_d to be equal to $\lambda_{d\ max}$ (the largest value possible if $s(r)$ is to be properly sampled). The program now returns to the section which assigns λ_d and puts $\lambda_d = \lambda_{d\ max}$. The constraint $\lambda_d \geq \lambda_{d\ min}$ is satisfied but the requirment that N_3 is sufficiently large for circular convolution is no longer true. Inspection of §2·545

E·4 shows that by reducing λ_d the value of N_3 for correct convolution has been increased. Thus with the initial choice $N_3 = 64$ the software is in a situation where the natural value of λ_d (determined by the data statistics) is too large, i.e. s(r) will be too narrow and therefore insufficiently sampled. Reduction of λ_d to $\lambda_{d\ max}$ (the maximum value for correct sampling of s) leads to a broadening of s which then violates the requirement that s and p must have sufficient zeros for the linear convolution to be correctly approximated by a circular convolution. The software therefore makes the only possible choice and starts again with a larger value of N_3 (the value predicted by §2·545 E·4). The software now starts again by calculating λ_d. Once again it passes the $\lambda_{d\ min}$ and N_3 checks, but fails the $\lambda_{d\ max}$ check (inevitably since $\lambda_d \leq \lambda_{d\ max}$ is true (or false) for all N_3, see §2·5421). Again λ_d is assigned the value $\lambda_{d\ max}$ and the checks repeated but this time with success because of the larger N_3. The program finishes by calculating $s * p$ for this value of λ.

```
PARAMETER LIST FOR FILE                       Z0011_PTRIM
------------------------------------------------------------
------------------------------------------------------------

LISTPARM REVISION NUMBER                      05 00
TRIM REVISION NUMBER                          00


DATA COLLECTED BY SH REVISION                 E
NUMBER OF LINES IN FILE DA0011                11
STEPPING MOTOR FREQUENCY                      500   STEPS PER SEC.
GATE PULSE SCALE FACTOR                       32
COLIMATOR DIAMETER : SOURCE                   5    MM
                     DETECTOR                 10    MM
NUMBER OF BLANK LINES FOR CALCULATING L       2
DIAMETER OF PHANTOM                           125    MM
NUMBER OF CELLS PER LINE                      52




PARAMETER LIST FOR FILE                       Z0011_PM0
----------------------------------------------------------
----------------------------------------------------------

LISTPARM REVISION NUMBER                      05 00.
M0 REVISION NUMBER                            00


SCANNER PARAMETERS
------------------

DELTA r, CELL SIZE                            6.502@ -1 CM
DELTA T, COUNTING TIME PER CELL               1.024@  0 SECS
COUNT RATE IN AIR                             1.332@  5 COUNTS/SEC
                                           +- 9.351@  2 COUNTS/SEC (1 S.D.)
NORMALISED DIAM. OF DET. COL.                 1.538@  0

NEW DATA PARAMETERS
-------------------

N2, NO. OF CELLS/LINE IN NEW DATA SET         64
M, NO. OF LINES IN NEW DATA SET               9

L PARAMETERS
------------

L, NO. OF COUNTS/CELL IN AIR                  1.364@  5
                                           +- 9.575@  2 (1 S.D.)
N1, NO. CELLS USED TO CALCULATE L             104
THETA, REL.ERROR IN L (3S CONF)               1.593@ -3
```

```
PARAMETER LIST FOR FILE                        Z0011_PGQ
--------------------------------------------------------
--------------------------------------------------------

LISTPARM REVISION NUMBER                       05 01
GQ REVISION NUMBER                             01


DATA PARAMETERS
---------------

N2, NO. OF CELLS/LINE IN NEW DATA SET          64
M, NO. OF LINES IN NEW DATA SET                9
DELTA r, CELL SIZE                             6.502@ -1 CM
DELTA R, FREQUENCY INCREAMENT                  2.403@ -2 CYCLES/CM

GAMMA PARAMETERS
----------------

MEAN                                           7.917@ -3
STANDARD DEVIATION                             9.328@ -4
MINIMUM                                        6.402@ -3
MAXIMUM                                        8.535@ -3

CONTINUOUS CUT-OFF FREQUENCY, OMEGA C COL
-----------------------------------------

MEAN                                           5.847@ -1
STANDARD DEVIATION                             1.201@ -2
MINIMUM                                        5.767@ -1
MAXIMUM                                        6.007@ -1

DISCRETE CUT-OFF FREQUENCY, K
-----------------------------

MEAN                                           2.433@  1
STANDARD DEVIATION                             5.000@ -1
MINIMUM                                        2.400@  1
MAXIMUM                                        2.500@  1
```

DETAILED LISTING OF PARAMETERS FOR FILE Z0011_PGQ

	REGULARISATION CONSTANT GAMMA	CONTINUOUS CUT-OFF FREQ. OMEGA C COL	DISCRETE CUT-OFF FREQ. K
LINE 1	6.828@ -3	6.007@ -1	25
LINE 2	6.828@ -3	6.007@ -1	25
LINE 3	8.535@ -3	5.767@ -1	24
LINE 4	8.535@ -3	5.767@ -1	24
LINE 5	8.535@ -3	5.767@ -1	24
LINE 6	8.535@ -3	5.767@ -1	24
LINE 7	8.535@ -3	5.767@ -1	24
LINE 8	8.535@ -3	5.767@ -1	24
LINE 9	6.402@ -3	6.007@ -1	25

```
PARAMETER LIST FOR FILE                       Z0011_PQP
-------------------------------------------------------
-------------------------------------------------------

LISTPARM REVISION NUMBER                      05 00
QP REVISION NUMBER                            01


DATA PARAMETERS
---------------

N2, NO. OF CELLS/LINE IN NEW DATA SET         64
M, NO. OF LINES IN NEW DATA SET               9

SMALL VALUE PARAMETERS
----------------------

MINIMUM NUMBER IN ANY LINE                    0
MAXIMUM NUMBER IN ANY LINE                    0
```

```
PARAMETER LIST FOR FILE                         Z0011_PPSP
--------------------------------------------------------
--------------------------------------------------------

LISTPARM REVISION NUMBER                        05 00
PSP REVISION NUMBER                             00


PARAMETER LIST FOR MAIN SECTION OF PSP : PASS     2
------------------------------------------------------

LITN, FILTER TYPE                               5
N, NO. OF CELLS/LINE IN M DATA                  32
N2, NO. OF CELLS/LINE IN P DATA                 64
DELTA r, CELL SIZE                              6.5020 -1 CM
N3, NO. OF CELLS/LINE IN s*p DATA               128
DELTA R, FREQUENCY INCRMNT IN s*p DATA          1.2010 -2 CYCLES/CM


PARAMETER LIST FOR LAMDA SECTION OF PSP : PASS     0
------------------------------------------------------

FILTER PARAMETERS :
n LAMDA d                                       7.3400  4
n LAMDA c                                       1.8380 -5
OMEGA c fil, CONTIN. CUT-OFF FREQ.              9.4690 -2 CYCLES/CM
K, DISCRETE CUT-OFF FREQ.                       7

n LAMDA d  SUPPLIED BY USER
           OR SET EQUAL TO n LAMDA d max
M, NO. OF LINES IN SCAN                         9
EQUIVALENT VALUE OF ESQD                        1.3230  0
EQUIVALENT VALUE OF ESQC                        6.0040 -1


PARAMETER LIST FOR MIN SECTION OF PSP : PASS     0
------------------------------------------------------

EXTREME PARAMETERS:
MAXERR, MAX. VALUE OF ALIASING ERROR
  ALLOWED IN s (SET BY USER)                    5.0000  0 % OF s
n LAMDA d min, MIN. ALLOWED VALUE OF
  LAMDA IF MAXERR IS NOT TO BE EXCEEDED         4.8790  0
OMEGA c fil min, CORRESPONDING VALUE
  OF CONTIN. CUT-OFF FREQ.                      1.3830 -2

PREDICTED PARAMETERS:
ERROR IN s ASSOCIATED WITH VALUE
  OF  n LAMDA d                                 2.5460 -1 % OF s
PREDICTED MIN. VALUE OF N3 SUCH THAT
  n LAMDA d >= n LAMDA d min                    71
PRACTICAL PREDICTED VALUE OF N3,
  PRED N32                                      128

TEST  N3 >= PRED N32:
RESULT                                          SUCCEED
```

```
PARAMETER LIST FOR MAX SECTION OF PSP : PASS     1
--------------------------------------------------

EXTREME PARAMETERS:
m, MINIMUM NO. OF SAMPLES ALLOWED
  IN FIRST PEAK OF s                          5.0
n LAMDA d max, MAX. VALUE OF LAMDA FOR
  PROPER SAMPLING OF s                        7.341@  4
OMEGA c fil max, CORRESPONDING VALUE
  OF CONTIN. CUT-OFF FREQ.                    9.469@ -2

PREDICTED PARAMETERS:
NO. OF SAMPLES IN FIRST PEAK OF s
  (PREDICTED VALUE OF m)                      5.0@  0
MIN N32, MIN. VALUE OF N3 SUCH THAT
  n LAMDA d max >= n LAMDA d min              32

TEST  n LAMDA d <= n LAMDA d max:
RESULT                                        SUCCEED
TEST  N3 >= MIN N32:
RESULT                                        SUCCEED
```

```
PARAMETER LIST FOR FILE                          Z0011_PPSPHIST
----------------------------------------------------------------
----------------------------------------------------------------

LISTPARM REVISION NUMBER                         05 01
PSP REVISION NUMBER                              00


PARAMETER LIST FOR MAIN SECTION OF PSP : PASS     1
-----------------------------------------------------

LITN, FILTER TYPE                                5
N, NO. OF CELLS/LINE IN m DATA                   52
N2, NO. OF CELLS/LINE IN P DATA                  64
DELTA r, CELL SIZE                               6.502@ -1 CM
N3, NO. OF CELLS/LINE IN s*p DATA                64
DELTA K, FREQUENCY INCRMNT IN s*p DATA   2.403@ -2 CYCLES/CM


PARAMETER LIST FOR LAMDA SECTION OF PSP : PASS     1
------------------------------------------------------

FILTER PARAMETERS :
n LAMDA d                                        2.831@  7
n LAMDA c                                        2.268@ -1
OMEGA c fil, CONTIN. CUT-OFF FREQ.               6.231@ -1 CYCLES/CM
K, DISCRETE CUT-OFF FREQ.                        25

n LAMDA d  CALCULATED  -  LINES CHOOSEN BY PROGRAM:
M, NO. OF LINES IN SCAN                          9
NO. OF LINES USED TO CALC. n LAMDA d             2
LINES USED IN CALCULATING  n LAMDA d
             1     6

ESQD  -  CALCULATED FROM DATA STATISTICS:
ESQD                                             1.954@ -3
TOL                                              5.0@  0 % OF ESQD


PARAMETER LIST FOR MIN SECTION OF PSP : PASS     1
----------------------------------------------------

EXTREME PARAMETERS:
MAXERR, MAX. VALUE OF ALIASING ERROR
  ALLOWED IN s (SET BY USER)                     5.000@  0 % OF s
n LAMDA d min, MIN. ALLOWED VALUE OF
  LAMDA IF MAXERR IS NOT TO BE EXCEEDED          4.879@  0
OMEGA c fil min, CORRESPONDING VALUE
  OF CONTIN. CUT-OFF FREQ.                       2.766@ -2

PREDICTED PARAMETERS:
ERROR IN s ASSOCIATED WITH VALUE
  OF  n LAMDA d                                  NEGLIGABLE
PREDICTED MIN. VALUE OF N3 SUCH THAT
  n LAMDA d >= n LAMDA d min                     55
PRACTICAL PREDICTED VALUE OF N3,
  PRED N32                                       64

TEST  N3 >= PRED N32:
RESULT                                           SUCCEED
```

```
PARAMETER LIST FOR MAX SECTION OF PSP : PASS      1
-------------------------------------------------

EXTREME PARAMETERS:
m, MINIMUM NO. OF SAMPLES ALLOWED
  IN FIRST PEAK OF s                          5.0
n LAMDA d max, MAX. VALUE OF LAMDA FOR
  PROPER SAMPLING OF s                        2.294@  3
OMEGA c fil max, CORRESPONDING VALUE
  OF CONTIN. CUT-OFF FREQ.                    9.469@ -2

PREDICTED PARAMETERS:
NO. OF SAMPLES IN FIRST PEAK OF s
  (PREDICTED VALUE OF m)                      1.6@  0
MIN N32, MIN. VALUE OF N3 SUCH THAT
  n LAMDA d max >= n LAMDA d min              32

TEST  n LAMDA d <= n LAMDA d max:
RESULT                                        FAIL
TEST  N3 >= MIN N32:
RESULT                                        SUCCEED


PARAMETER LIST FOR LAMDA SECTION OF PSP : PASS      1
---------------------------------------------------

FILTER PARAMETERS :
n LAMDA d                                     2.294@  3
n LAMDA c                                     1.838@ -5
OMEGA c fil, CONTIN. CUT-OFF FREQ.            9.469@ -2 CYCLES/CM
K, DISCRETE CUT-OFF FREQ.                     3

n LAMDA d  SUPPLIED BY USER
           OR SET EQUAL TO n LAMDA d max
M, NO. OF LINES IN SCAN                       9
EQUIVALENT VALUE OF ESQD                      1.322@  0
EQUIVALENT VALUE OF ESQC                      6.003@ -1


PARAMETER LIST FOR MIN SECTION OF PSP : PASS      1
-------------------------------------------------

EXTREME PARAMETERS:
MAXERR, MAX. VALUE OF ALIASING ERROR
  ALLOWED IN s (SET BY USER)                  5.000@  0 % OF s
n LAMDA d min, MIN. ALLOWED VALUE OF
  LAMDA IF MAXERR IS NOT TO BE EXCEEDED       4.879@  0
OMEGA c fil min, CORRESPONDING VALUE
  OF CONTIN. CUT-OFF FREQ.                    2.766@ -2

PREDICTED PARAMETERS:
ERROR IN s ASSOCIATED WITH VALUE
  OF  n LAMDA d                               1.007@  0 % OF s
PREDICTED MIN. VALUE OF N3 SUCH THAT
  n LAMDA d >= n LAMDA d min                  71
PRACTICAL PREDICTED VALUE OF N3,
  PRED N32                                    123

TEST  N3 >= PRED N32:
RESULT                                        FAIL
```

```
PARAMETER LIST FOR MAIN SECTION OF PSP : PASS     2
-------------------------------------------------

LITN, FILTER TYPE                             5
N, NO. OF CELLS/LINE IN m DATA                52
N2, NO. OF CELLS/LINE IN P DATA               64
DELTA r, CELL SIZE                            6.502@ -1 CM
N3, NO. OF CELLS/LINE IN s*p DATA             128
DELTA K, FREQUENCY INCRMNT IN s*p DATA        1.201@ -2 CYCLES/CM


PARAMETER LIST FOR LAMDA SECTION OF PSP : PASS     1
--------------------------------------------------

FILTER PARAMETERS :
n LAMDA d                                     9.060@  0
n LAMDA c                                     2.268@ -1
OMEGA c fil, CONTIN. CUT-OFF FREQ.            6.231@ -1 CYCLES/CM
K, DISCRETE CUT-OFF FREQ.                     51

n LAMDA d  CALCULATED  -  LINES CHOOSEN BY PROGRAM:
M, NO. OF LINES IN SCAN                       9
NO. OF LINES USED TO CALC. n LAMDA d          2
LINES USED IN CALCULATING  n LAMDA d
            1    6

ESQD  -  CALCULATED FROM DATA STATISTICS:
ESQD                                          1.754@ -3
TOL                                           5.0@  0 % OF ESQD


PARAMETER LIST FOR MIN SECTION OF PSP : PASS     1
------------------------------------------------

EXTREME PARAMETERS:
MAXERR, MAX. VALUE OF ALIASING ERROR
  ALLOWED IN s (SET BY USER)                  5.000@  0 % OF s
n LAMDA d min, MIN. ALLOWED VALUE OF
  LAMDA IF MAXERR IS NOT TO BE EXCEEDED       4.379@  0
OMEGA c fil min, CORRESPONDING VALUE
  OF CONTIN. CUT-OFF FREQ.                    1.383@ -2

PREDICTED PARAMETERS:
ERROR IN s ASSOCIATED WITH VALUE
  OF  n LAMDA d                               NEGLIGABLE
PREDICTED MIN. VALUE OF N3 SUCH THAT
  n LAMDA d >= n LAMDA d min                  55
PRACTICAL PREDICTED VALUE OF N3,
  PRED N32                                    64

TEST  N3 >= PRED N32:
RESULT                                        SUCCEED
```

```
PARAMETER LIST FOR MAX SECTION OF PSP : PASS      1
-------------------------------------------------

EXTREME PARAMETERS:
a, MINIMUM NO. OF SAMPLES ALLOWED
  IN FIRST PEAK OF s                          5.0
n LAMDA d max, MAX. VALUE OF LAMDA FOR
  PROPER SAMPLING OF s                        7.341@  4
OMEGA c fil max, CORRESPONDING VALUE
  OF CONTIN. CUT-OFF FREQ.                    9.469@ -2

PREDICTED PARAMETERS:
NO. OF SAMPLES IN FIRST PEAK OF s
  (PREDICTED VALUE OF a)                      1.60  0
MIN N32, MIN. VALUE OF N3 SUCH THAT
  n LAMDA d max >= n LAMDA d min              32

TEST  n LAMDA d <= n LAMDA d max:
RESULT                                        FAIL
TEST  N3 >= MIN N32:
RESULT                                        SUCCEED


PARAMETER LIST FOR LAMDA SECTION OF PSP : PASS      1
---------------------------------------------------

FILTER PARAMETERS :
n LAMDA d                                     7.340@  4
n LAMDA c                                     1.838@ -5
OMEGA c fil, CONTIN. CUT-OFF FREQ.            9.469@ -2 CYCLES/CM
K, DISCRETE CUT-OFF FREQ.                     7

n LAMDA d  SUPPLIED BY USER
           OR SET EQUAL TO n LAMDA d max
M, NO. OF LINES IN SCAN                       9
EQUIVALENT VALUE OF ESQD                      1.323@  0
EQUIVALENT VALUE OF ESQC                      6.004@ -1


PARAMETER LIST FOR MIN SECTION OF PSP : PASS      1
-------------------------------------------------

EXTREME PARAMETERS:
MAXERR, MAX. VALUE OF ALIASING ERROR
  ALLOWED IN s (SET BY USER)                  5.000@  0 % OF s
n LAMDA d min, MIN. ALLOWED VALUE OF
  LAMDA IF MAXERR IS NOT TO BE EXCEEDED       4.879@  0
OMEGA c fil min, CORRESPONDING VALUE
  OF CONTIN. CUT-OFF FREQ.                    1.383@ -2

PREDICTED PARAMETERS:
ERROR IN s ASSOCIATED WITH VALUE
  OF  n LAMDA d                               2.546@ -1 % OF s
PREDICTED MIN. VALUE OF N3 SUCH THAT
  n LAMDA d >= n LAMDA d min                  71
PRACTICAL PREDICTED VALUE OF N3,
  PRED N32                                    128

TEST  N3 >= PRED N32:
RESULT                                        SUCCEED
```

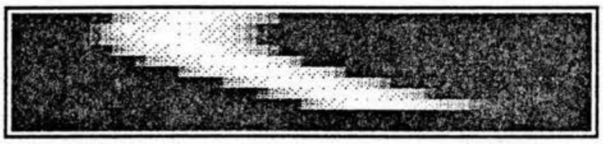
```
PARAMETER LIST FOR MAX SECTION OF PSP : PASS      1
-------------------------------------------------

EXTREME PARAMETERS:
m, MINIMUM NO. OF SAMPLES ALLOWED
  IN FIRST PEAK OF s                        5.0
n LAMDA d max, MAX. VALUE OF LAMDA FOR
  PROPER SAMPLING OF s                      7.341@  4
OMEGA c fil max, CORRESPONDING VALUE
  OF CONTIN. CUT-OFF FREQ.                  9.469@ -2

PREDICTED PARAMETERS:
NO. OF SAMPLES IN FIRST PEAK OF s
  (PREDICTED VALUE OF m)                    5.00@  0
MIN N32, MIN. VALUE OF N3 SUCH THAT
  n LAMDA d max >= n LAMDA d min            32

TEST  n LAMDA d <= n LAMDA d max:
RESULT                                      SUCCEED
TEST  N3 >= MIN N32:
RESULT                                      SUCCEED


PROGRAM STOPPED AT END
```

PARAMETER LIST FOR FILE	Z0011_PSPF
LISTPARM REVISION NUMBER	05 00
SPF REVISION NUMBER	00
CELL SIZE IN M-DATA	6.502@ -1 CMS
OMEGA c fil	9.469@ -2 CYCLES/CM
OMEGA c ang	2.292@ -1 CYCLES/CM
OMEGA c col	5.847@ -1 CYCLES/CM
CELL SIZE : CELL SIZE IN M DATA	6.502@ -1 CMS
NUMBER OF CELLS IN F MATRIX	64
TYPE OF INTERPOLATION FOR SP-DATA	STEP FN.
REMOVE -VE. VALUES IN F MATRIX?	NO

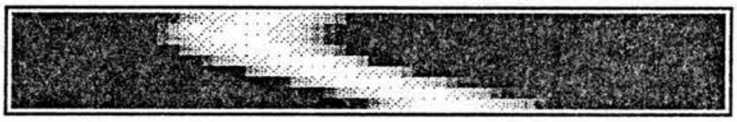
```
PRESET PARAMETER LIST FOR PROGRAM PSP
-------------------------------------
-------------------------------------


PARAMETER LIST FOR MAIN SECTION OF PSP :
----------------------------------------

LITN, FILTER TYPE                              5
N, NO. OF CELLS/LINE IN M DATA                 52
N2, NO. OF CELLS/LINE IN P DATA                64
DELTA r, CELL SIZE                             6.5020 -1 CM
N3, NO. OF CELLS/LINE IN s*p DATA              64


PARAMETER LIST FOR LAMDA SECTION OF PSP :
-----------------------------------------

n LAMDA d  CALCULATED  -  LINES CHOOSEN BY PROGRAM:
M, NO. OF LINES IN SCAN                        9
NO. OF LINES USED TO CALC. n LAMDA d           2
LINES USED IN CALCULATING  n LAMDA d
            1    6

ESQD  -  CALCULATED FROM DATA STATISTICS:
TOL                                            5.00  0 % OF ESQD


PARAMETER LIST FOR MIN SECTION OF PSP :
---------------------------------------

MAXERR, MAX. VALUE OF ALIASING ERROR
  ALLOWED IN s (SET BY USER)                   5.0000  0 % OF s

NEXT STEP IF  LAMDA >= LAMDA min:
  CONTINUE ( CHECK LAMDA <= LAMDA max )

NEXT STEP IF  LAMDA < LAMDA min:
  USE PREDICTED NO. OF DATA POINTS AND RECALCULATE
  LAMDA USING SAME DATA LINES


PARAMETER LIST FOR MAX SECTION OF PSP :
---------------------------------------

ACTION IF LAMDA max < LAMDA min
  ADJUST N3 SO THAT LAMDA max >= LAMDA min

m, MIN. ALLOWED NO. OF SAMPLES IN FIRST PEAK OF s   5.00  0

NEXT ACTION IF LAMDA<=LAMDA max
  CONTINUE (CALCULATE s*p AND TIDY UP FILES)

NEXT ACTION IF LAMDA>LAMDA max
  CALCULATE s*p WITH LAMDA=LAMDA max


ACTION ON REACHING END OF PROGRAM PSP :
---------------------------------------

  RECALC. LAMDA BY SAME METHOD AS PREVIOUSLY BUT
  USING ALL DATA LINES
```

PRESET PARAMETER LIST FOR PROGRAM SPF

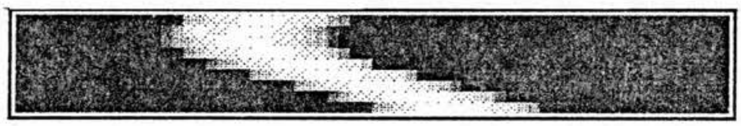
CELL SIZE : CELL SIZE IN M DATA	
NUMBER OF CELLS IN F MATRIX	64
TYPE OF INTERPOLATION FOR SP-DATA	STEP FN.
REMOVE -VE. VALUES IN F MATRIX?	NO



Z0011*ESTM
MIN= 5.830e 4   MAX= 1.407e 5
01/05/80   20.52.03



Z0011*ESTG
MIN= 4.274e -1   MAX= 1.031e 0
01/05/80   20.49.07



Z0011*ESTQ
MIN= 4.251e -1   MAX= 1.045e 0
01/05/80   20.49.28

fig. 6·1a : Example of data files.

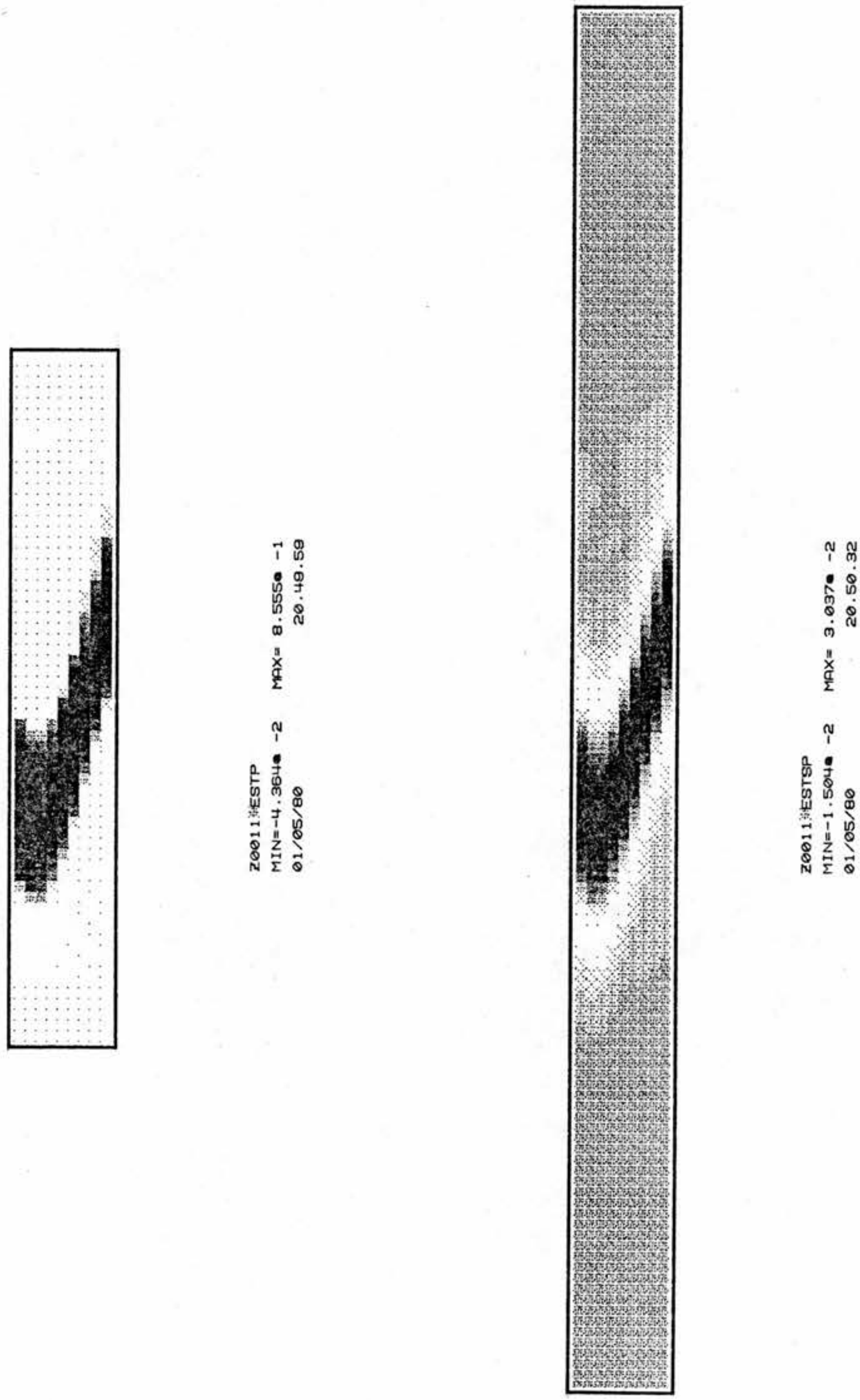


fig. 6·1b : Example of data files.

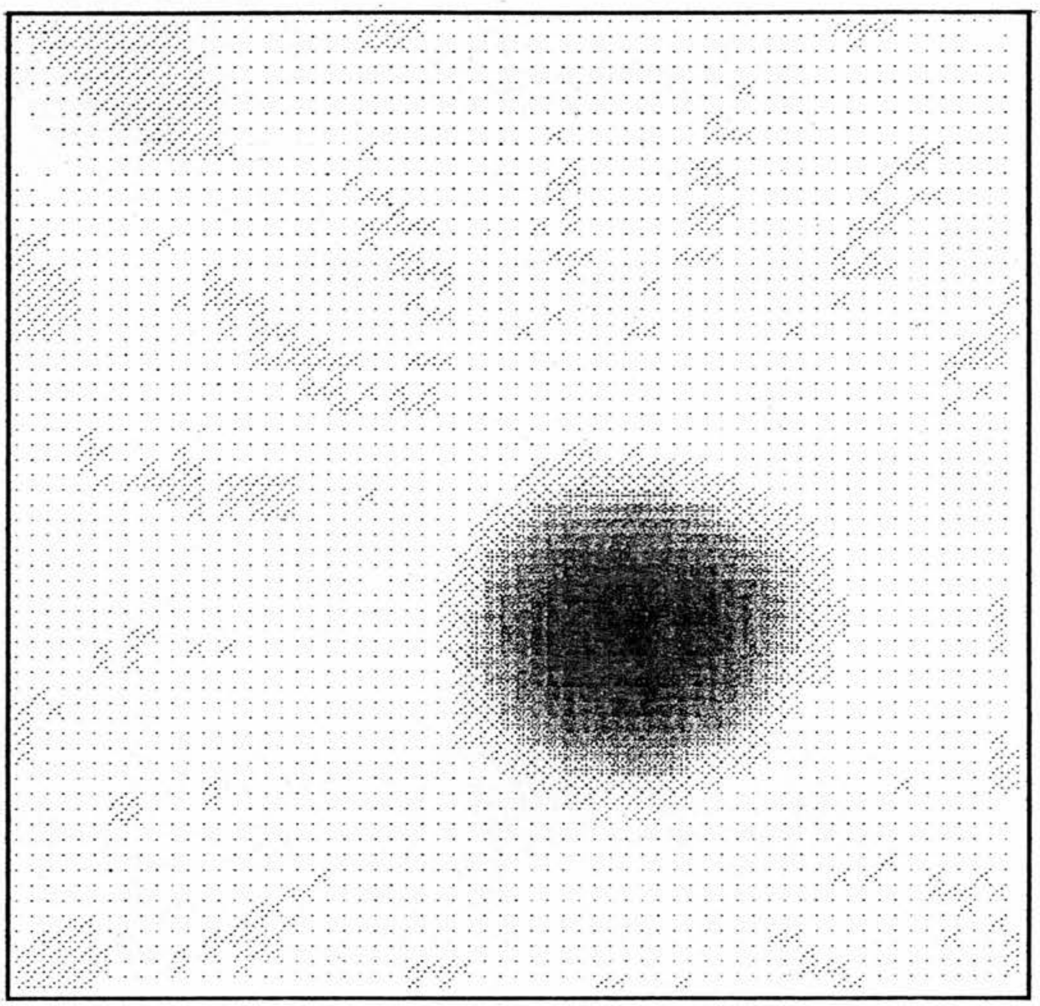


fig. 6·1c : Example of data files.

·3 Problems

·31 Collimater allignment

It was immediately clear from visual observation of the count display on the scanner that the number of counts/cell was varying too much even when there was no object between source and detector.

A worst case situation (source and detector both with circular collimation of 1 mm diameter and 150 mm length) is illustrated in F·2 which shows a graph of counts/cell against cell position for two consecutive traverses. Even allowing for data statistics there is an obvious drop in count rate in the centre of each traverse. This problem was found to be caused by bending of the steel rods supporting the U-frame. The scanner was therefore modified by placing a steel track under each of the rods and using these to give additional upward thrust (via a suitable bearing) to the underside of the U-frame. This improved matters but it was clear that the only complete cure required a redesign of the whole scanner.

As this was not possible it was decided that only the 5 mm and 10 mm diameter collimaters would be used because in these cases the effect was relatively minor. Corresponding graphs for these cases are given in F·3a and F·3b respectively.

·32 Data errors

When the data from the scans of the various objects (see §·41) was processed various anomolous results were encountered. These were traced to occasional clearly erroneous count values in a few readings. A typical case is illustrated in F·4 which shows a projection of the water annulus (see §·41) before and after correction. The corrected reading was derived simply by inspecting the surrounding values and substituting one of similar size, a procedure considered to be justified by the nature of the scanning process. All the data was visually inspected in this way and corrections applied when necessary.

A more fundamental cure was not possible due to lack of time.

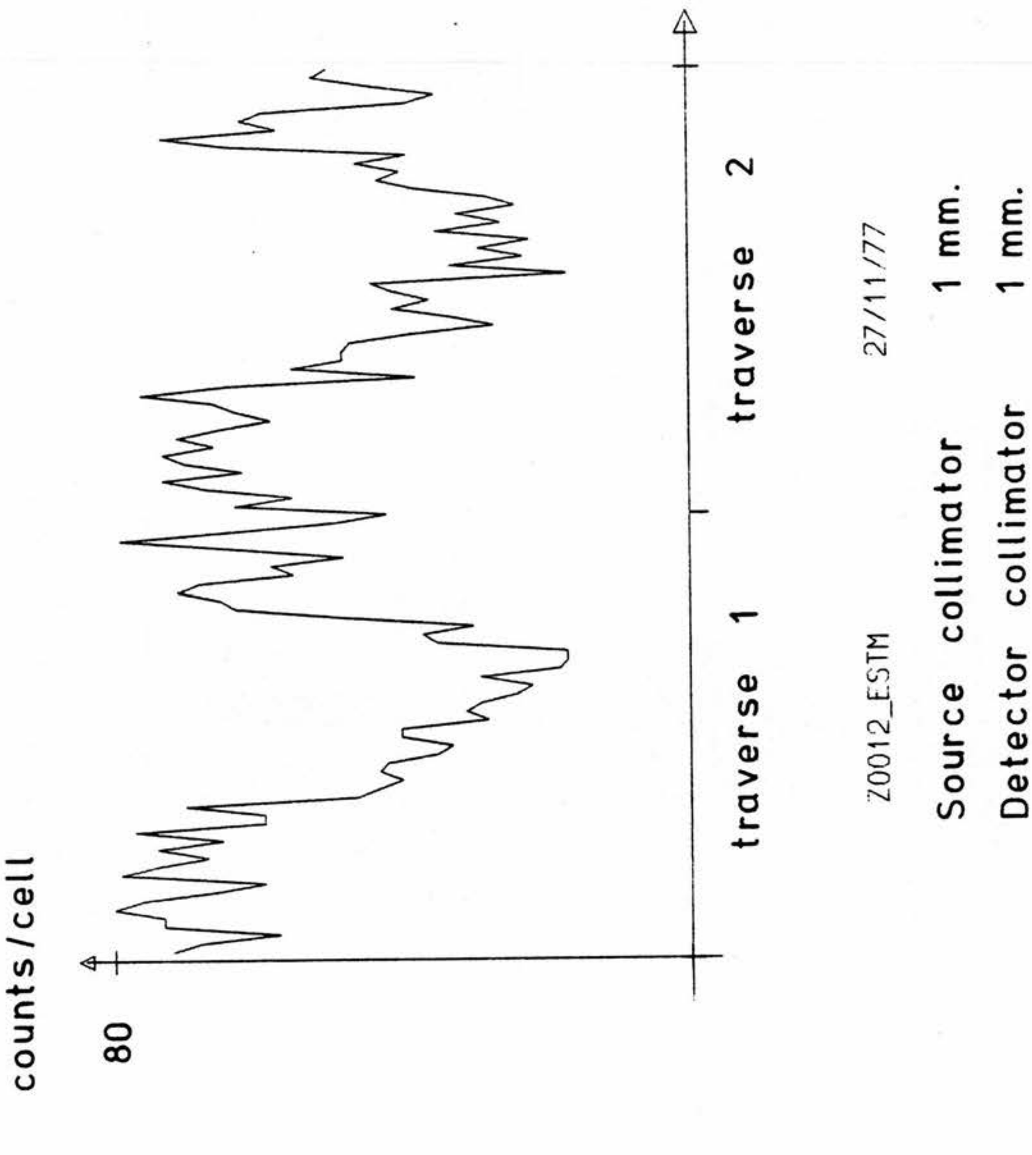


fig. 6·2 : Collimator allignment.

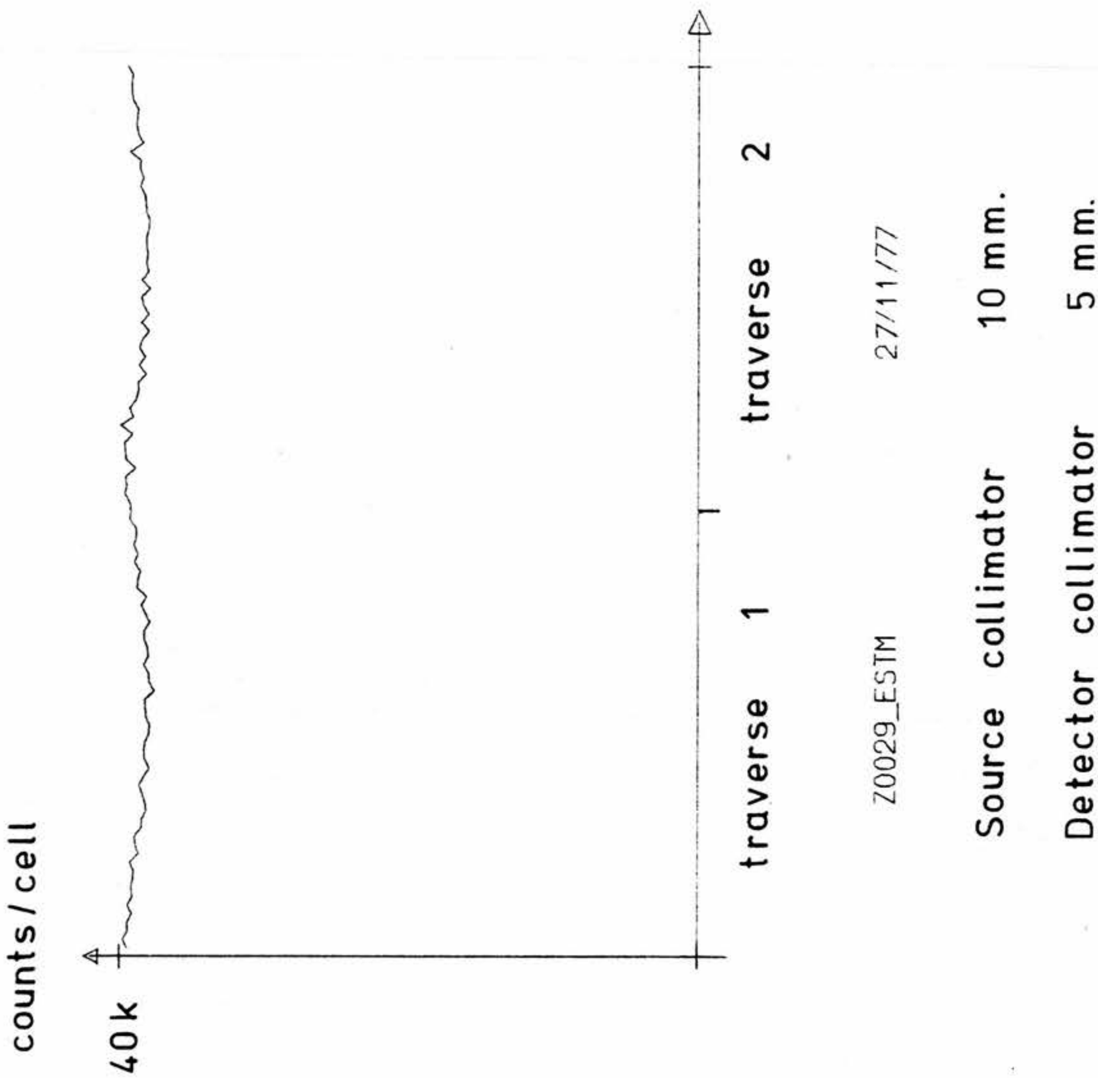


fig. 6·3a : Collimator allignment.

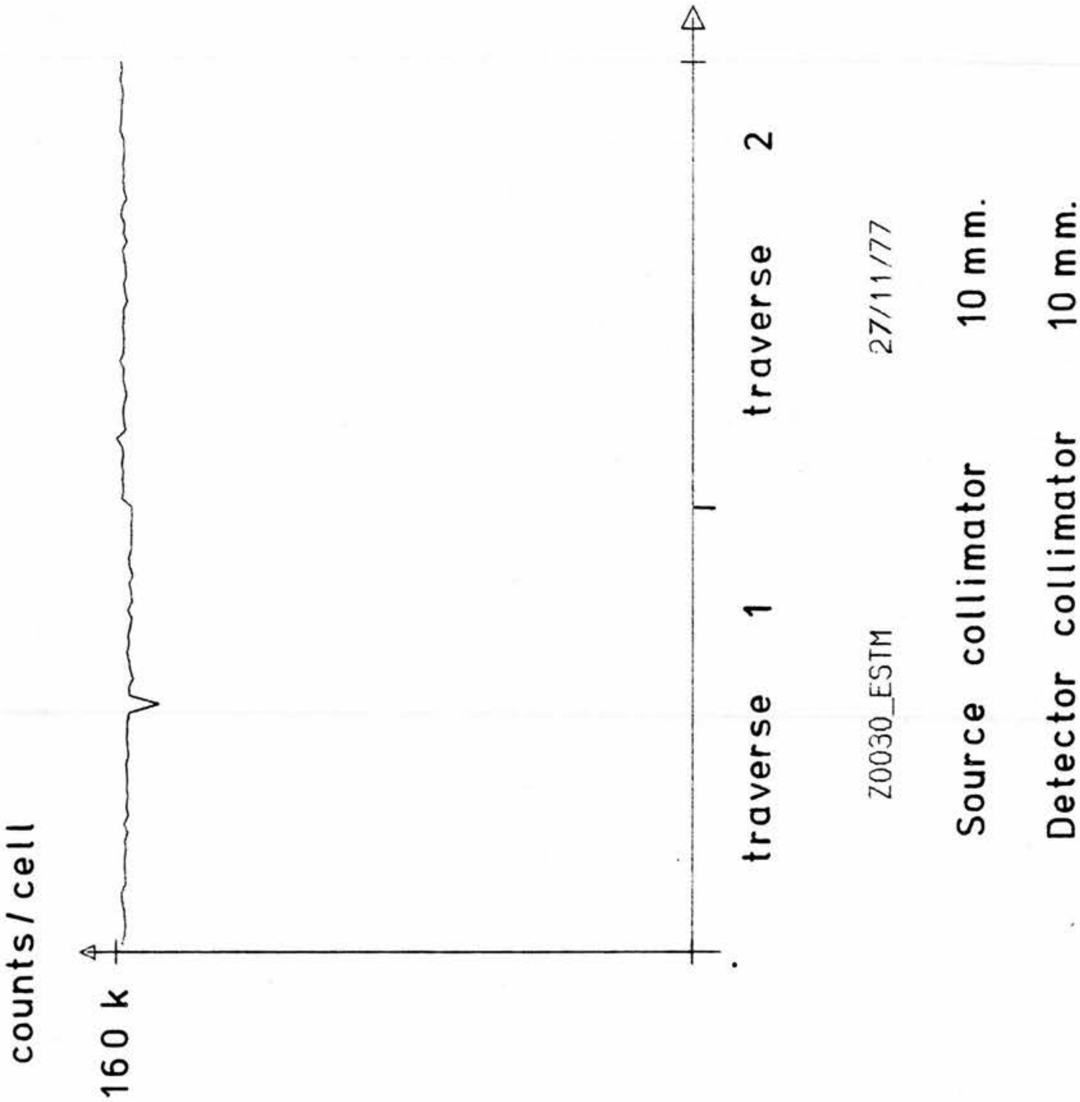
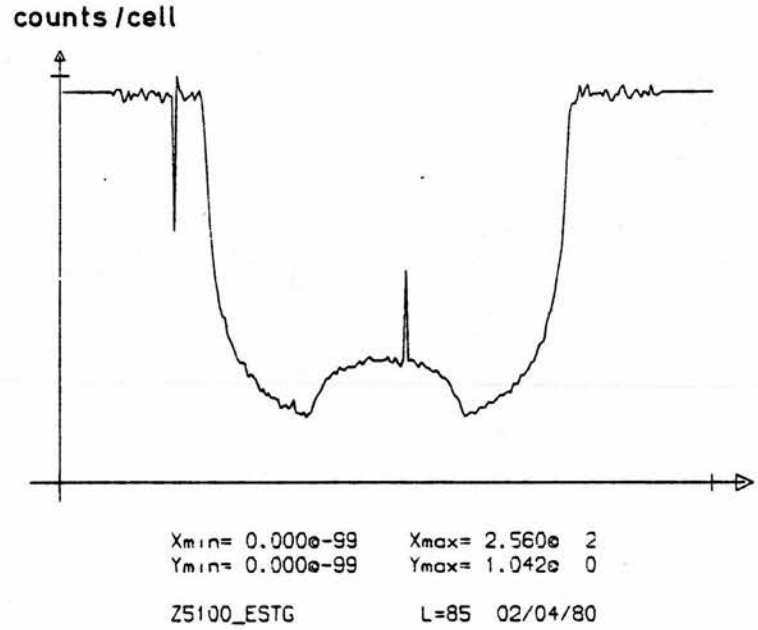
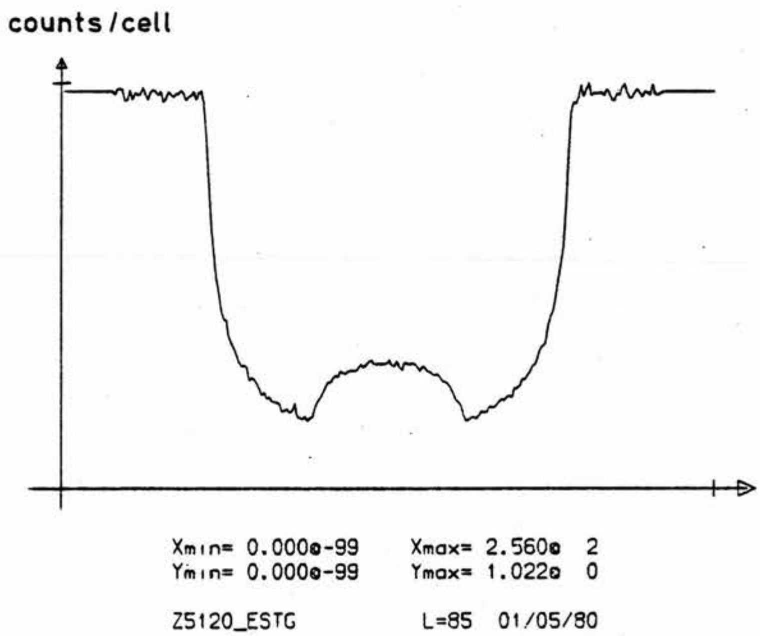


fig. 6·3 b : Collimator allignment.



Before data patching.



After data patching.

fig. 6·4 : Data patching.

·4 Assessment of mathematics and software

The aim of this section is to try and assess whether the mathematics and data processing worked properly. It had originally been intended that this whole assessment process would be much more comprehensive but lack of time necessitated a drastic pruning to the barest essentials.

The result of this cutting back was that four test objects were chosen (see §·41) and experiments performed on each in turn to assess whether:-

1) the reconstruction software chooses an $\hat{f}$ close to f,
2) the prediction software does in fact choose the optimum value of M,
3) the prediction software does in fact choose the optimum value of k,
4) the prediction software does in fact predict a value of ℓ which will give rise to the chosen value of $\omega_{c\ fil}$.

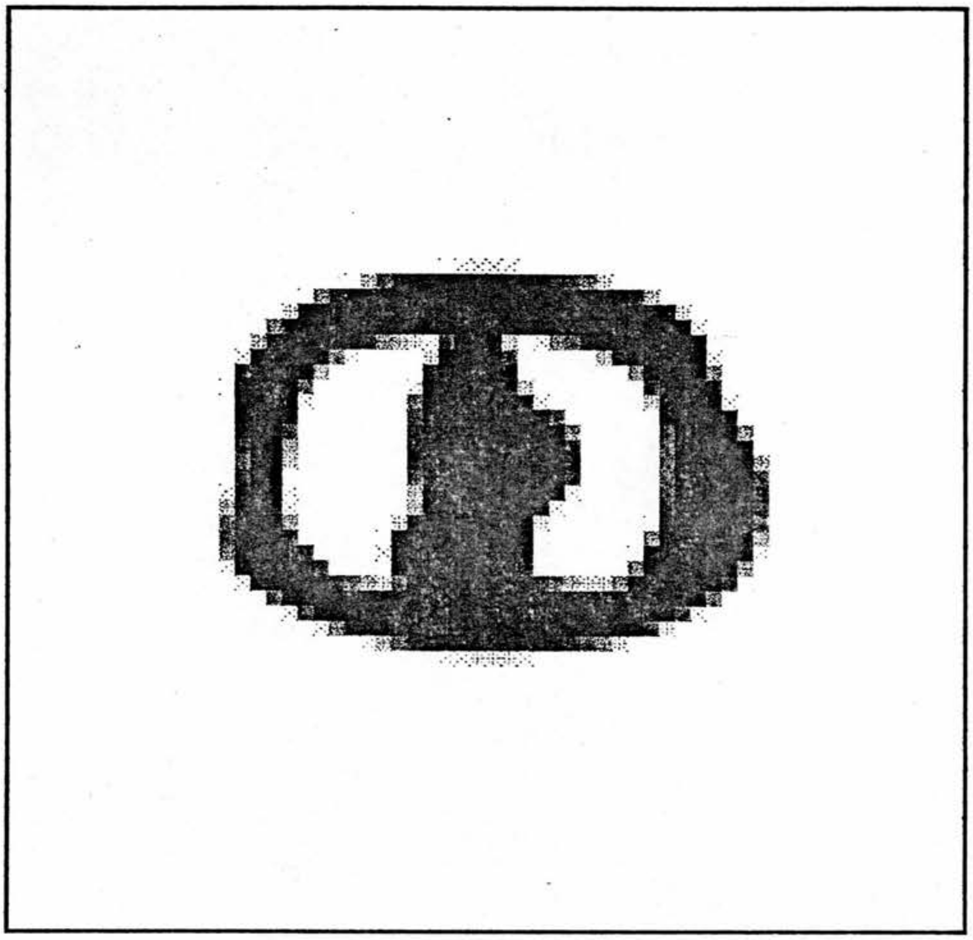
These matters are dealt with in §·42 - §·45 respectively. Note that in each of the above no deconvolution is performed (i.e. only a simulated run of GQ is performed, see §5·432), that step function interpolation has been used in SPF and that prior to evaluating $L_{2\ rel}$ (see §5·47) in COMPARE all negative values in f have been replaced by zero. This work is followed by information on:-

5) the effect of step function or linear interpolation
6) the sort of edge responses that were encountered
7) the effect of deconvolution

in §·46 - §·48.

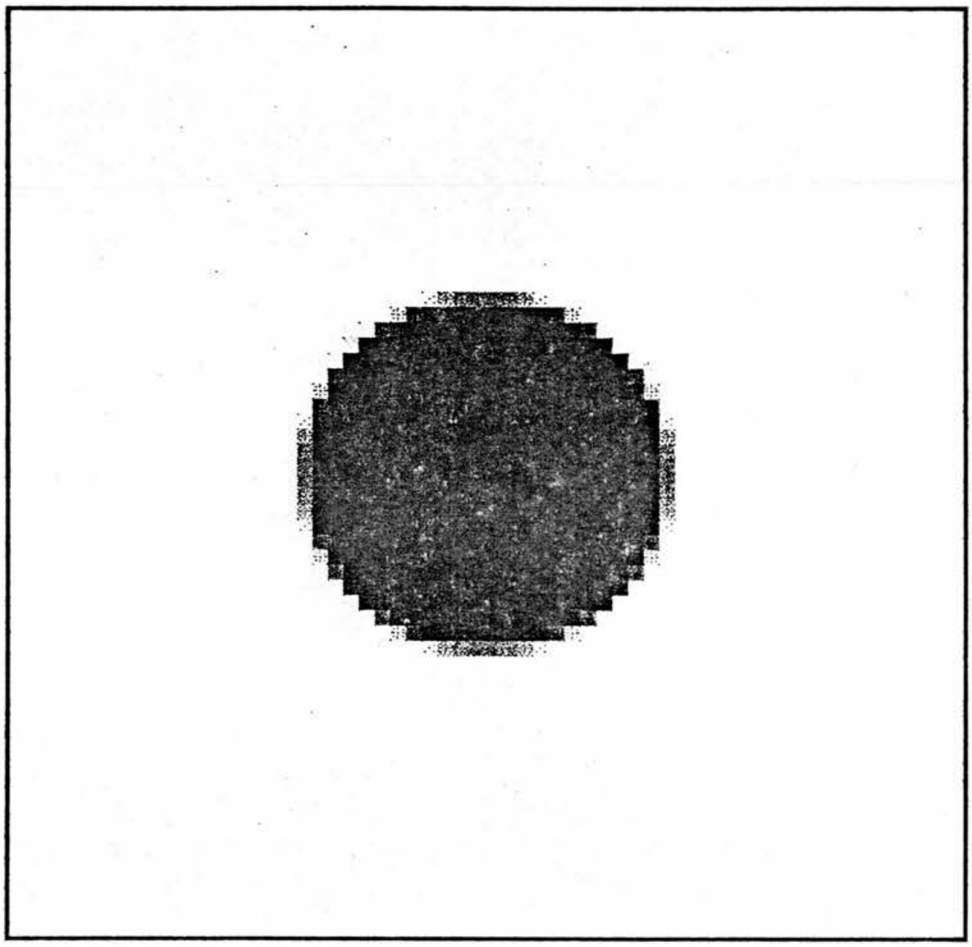
·41 Test objects

Four test objects were used and are illustrated in F·5 - F·8. These four greyscale plots are all on the same scale (a 60 × 60 matrix of 1 cm cells). The lung phantom (F·5) is made of wax (with a linear attenuation coefficient of 0·08263 cm^{-1}) and two air holes to represent lungs. The disc (F·6) is made of a polythene bucket full of water. The annulus (F·7) is the same as the



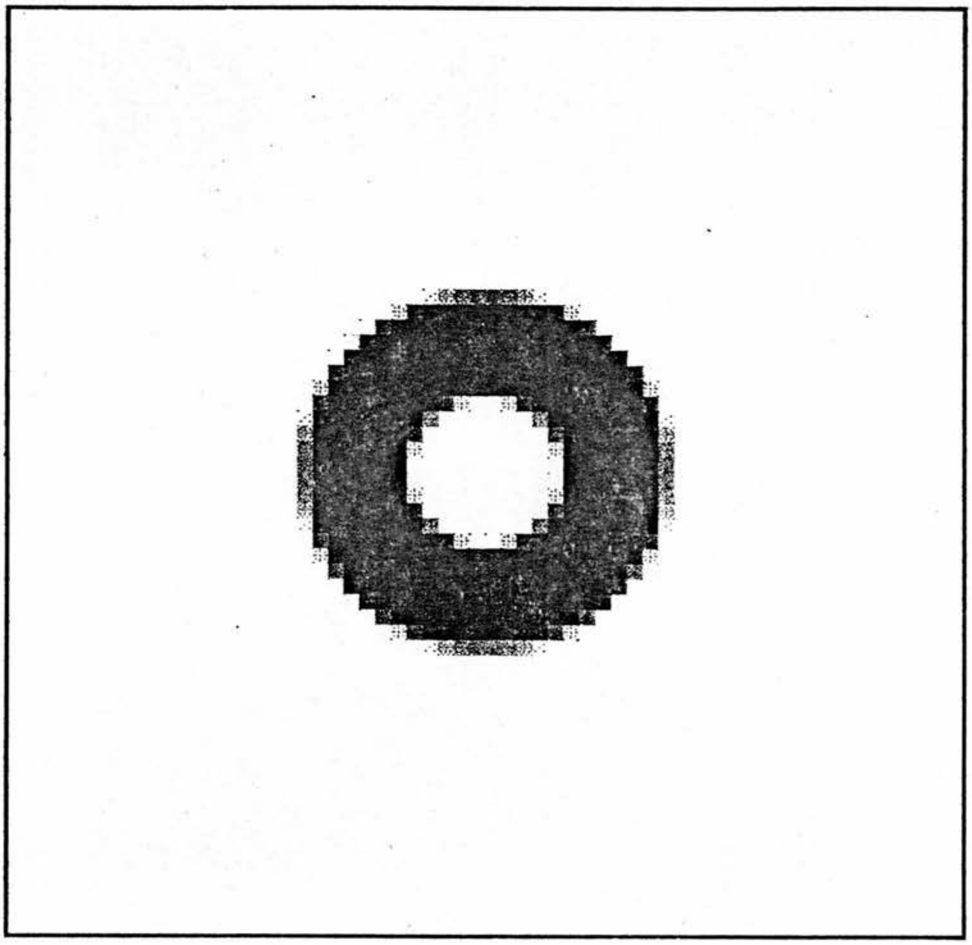
Z4540 PHANTOM
MIN= 0.000e-99   MAX= 8.263e -2
15/04/80   20.02.32

fig. 6·5 : Lung phantom.



Z4740 PHANTOM
MIN= 0.000@-99 MAX= 8.500@ -2
16/04/80 09.04.30

fig. 6·6 : Disc phantom.



Z4940 PHANTOM
MIN= 0.000e-99   MAX= 8.500e -2
16/04/80   09.04.55

fig. 6·7 : Annular phantom.

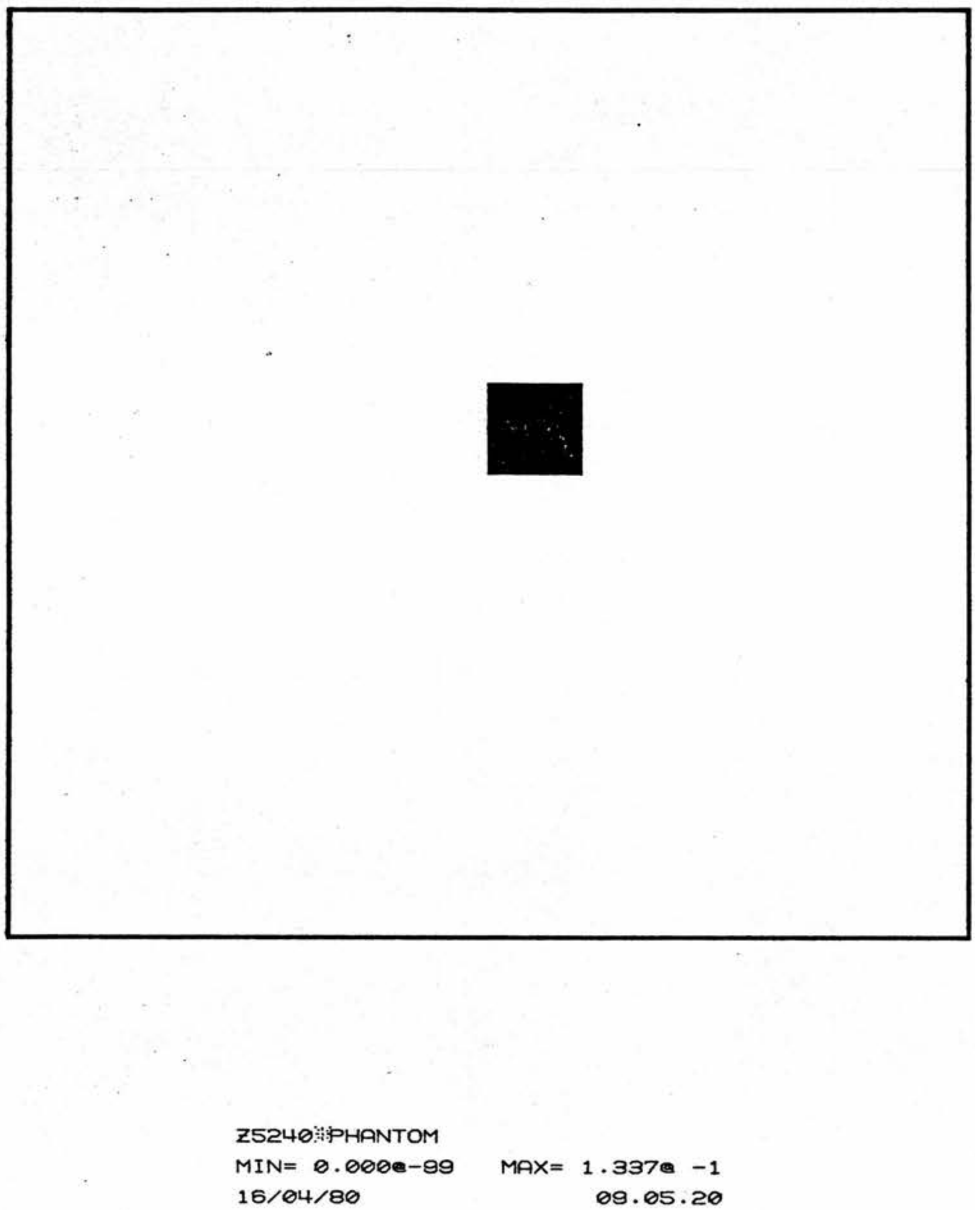


fig. 6·8 : Block phantom.

disc, but with a further container placed in the water to give the air hole. The rectangle (F·8) is a 6 cm square block of tufnol (with a linear attenuation coefficient of 0·1337 cm^{-1}) placed with its lower left corner at the centre of rotation.

When comparing these objects with their reconstructions given below, there are a few points to be borne in mind.

a) a scan length of 34 cms was used for the lung, the disc and the annulus and a scan length of 20 cms for the block.

b) when running the prediction programs the parameters calculated for setting up the scanner are calculated so that the required spatial resolution is obtained over a circle with the same diameter as the phantom and centred in the reconstruction rectangle. Artifacts outside this area should be ignored when considering the quality of a reconstruction. (see §3·43221 for definition of phantom diameter)

c) sections through the phantoms are taken from left to right along row 30 (counting from 1 at the bottom) for the lung, disc and annulus and along row 33 for the block.

d) prediction was based on the following matrix sizes:-

lung	36 × 36 matrix of 1 cm cells
disc	60 × 60 matrix of 0·5 cm cells
annulus	60 × 60 matrix of 0·5 cm cells
block	12 × 12 matrix of 1 cm cells

Calculation of $L_{2\ rel}$ by COMPARE was based on the same matrices except where otherwise stated.

Finally the reader is asked to note one qualification which applies to all the results concerning the block. The PREDICT program was run using a scan traverse length of 34 cm for all phantoms. When it came to performing the scans it was found that the number of cells/line exceeded 256 in the case of the block. Since 256 was the maximum allowed by SH1, the scans with the block were all performed with the scan length reduced to 20 cms and all other parameters as predicted. Inspection of §3·43225 shows that the effect of using PREDICT with an incorrectly large value of L is to increase S_1 and hence ℓ. Thus in effect the block was not scanned with the correct ℓ.

·42 Selection of $\omega_{c\ fil}$

·421 Aim and method

The value of $\omega_{c\ fil}$ is determined by §2·532 E·7 to be the value which minimises $\|\nabla^2 f\|_{B^2}^{2}$ subject to $d(\hat{p}, \mathcal{R} f) \le e$. The aim of this experiment is to examine whether the $\hat{f}$ obtained is actually close to f. The method was to choose $\omega_{c\ rec}$, use PREDICT to determine how each of the objects should be scanned and scan the object in this manner. The data was then reconstructed with lamda estimated according to §2·532 E·7 and $L_{2\ rel}$ calculated. This value of lamda is denoted by λ_c^*. The same data was then processed a further eleven times but with forced values of λ_c such that λ_c/λ_c^* equalled 0·01, 0·02, 0·05, 0·1, 0·2, 0·5, 2, 5, 10, 20, 50 and in each case $L_{2\ rel}$ was again evaluated.

·422 Scanning details

For each of the objects used in this experiment the following details are given in the table below:-

test object used

value of $\omega_{c\ rec}$ used in PREDICT

value of 2a, r, M, ℓ calculated by PREDICT

names of associated data files (which appear in the figures).

object	lung	disc	annulus	annulus	block
$\omega_{c\ rec}$ (cycles/cm)	0·500	0·500	1·000	0·500	0·500
2a (cm)	1·000	1·000	0 ·500	1·000	1·000
Δr (cm)	0 ·2154	0·1642	0·1710	0·1642	0·1231
M	59	38	76	38	25
ℓ (counts/cell)	4500	14000	25000	7800	4600
data files	Z4500 - Z4511	Z4720 - Z4731	Z4950 - Z4961	Z5020 - Z5031	Z5250 - Z5261

In each case the lowest number data file corresponds to $\lambda_c/\lambda_c^* = 1$ and the remaining eleven to $\lambda_c/\lambda_c^* = 0{\cdot}01, \ldots, \lambda_c/\lambda_c^* = 50$ respectively.

·423 Results

The resulting values of $L_{2\ rel}$ are graphed as functions of λ_c/λ_c^* in F·9 and the change produced in the image is illustrated for the lung phantom in F·10 - F·21.

It is clear that the value assigned by the software to λ_c is near the optimum in each case. (The author finds the anomolous behaviour of the tufnol block curve (Z52) disturbing but is unaware of anything incorrect in the data or its processing which would explain it other than the reservation expressed in section ·41.)

·43 Prediction of M

·431 Aim and method

The aim of this experiment was to test that the prediction software (in particular the criterion §3·421 E·12) does select an appropriate value for M. The method is based on a qualitative understanding of how $L_{2\ rel}$ will vary as M is altered. If M is too small then one would expect $L_{2\ rel}$ to increase because of the poor quality of the reconstructed image. As M increases one expects $L_{2\ rel}$ to decrease at first and then become stationery since, for large M, the image quality will be determined by $\omega_{c\ col}$ and $\omega_{c\ fil}$. The test of objects were therefore scanned with M = 128 and by repeatedly selecting only alternate projections it was possible to generate data corresponding to M = 64, 32, 16, 8, 4. In this way data was generated corresponding to scanning the test objects with various M. This data was processed and $L_{2\ rel}$ calculated in each case.

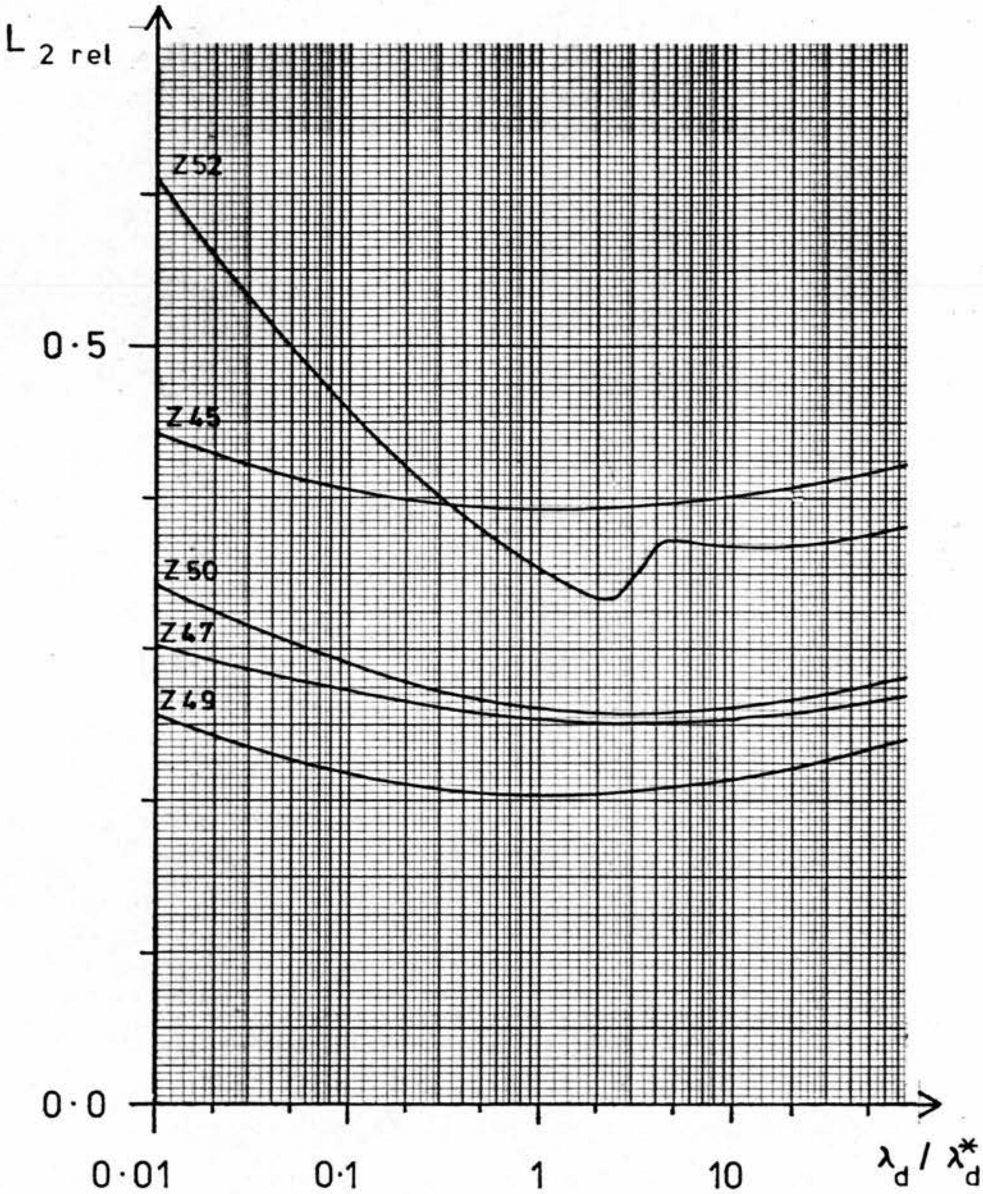
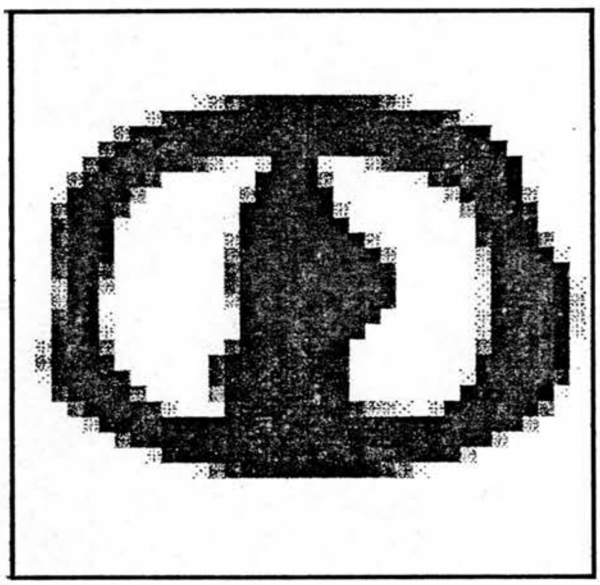
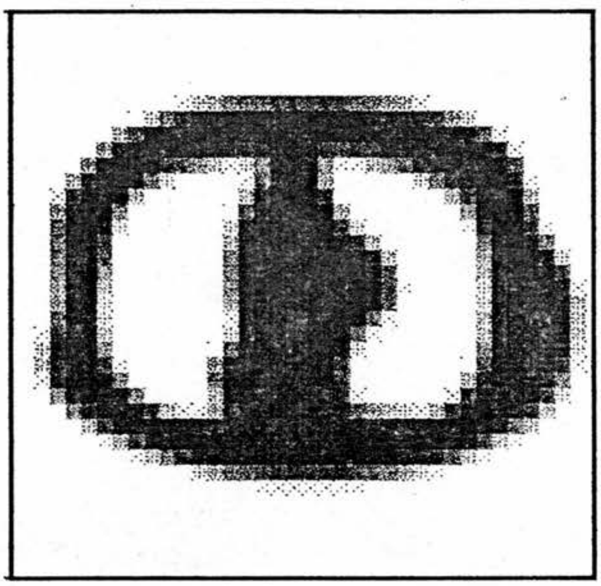


fig. 6·9 : Graph of $L_{2\,rel}(\lambda_d / \lambda_d^*)$.



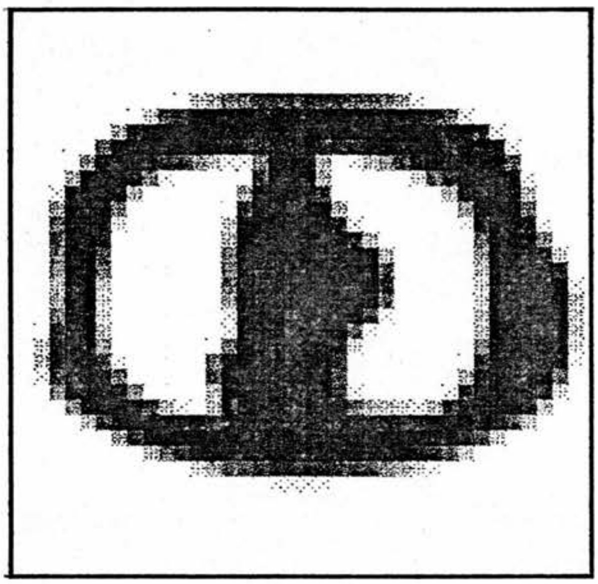
Z4500:ESTF
MIN= 0.000e-99 MAX= 7.897e -2
31/03/80 14.48.56

fig. 6·10 : Lung phantom, $\lambda_d = \lambda_d^*$.



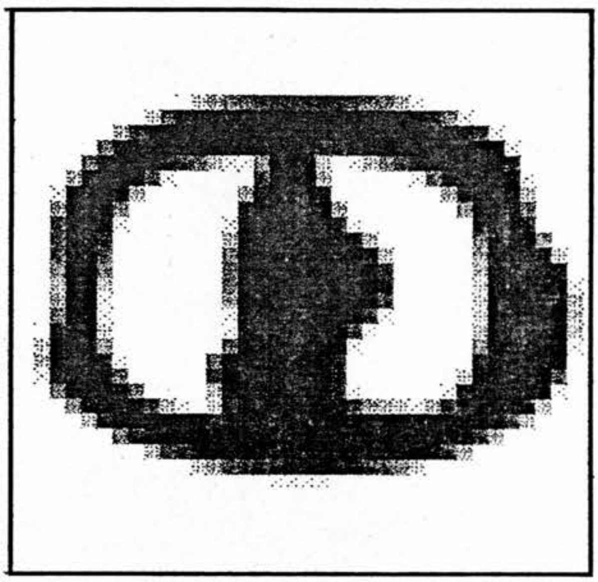
Z4501 ESTF
MIN= 0.000e-99   MAX= 7.504e -2
31/03/80   14.47.31

fig. 6·11 : Lung phantom , $\lambda_d = 0{\cdot}01\,\lambda_d^*$.



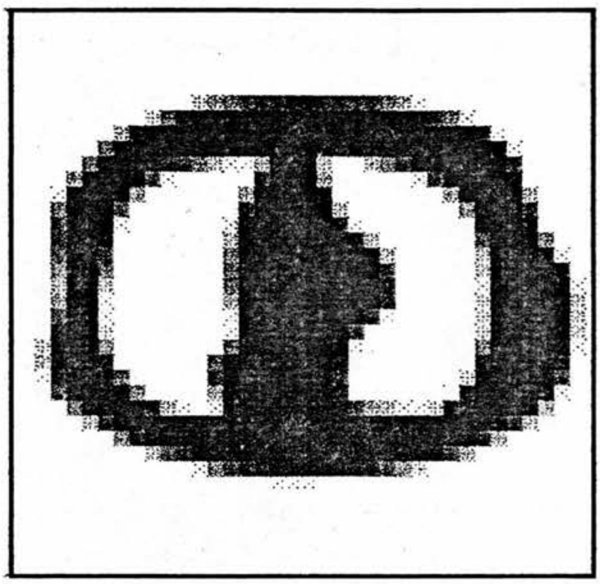
Z4502:ESTF
MIN= 0.000e-99   MAX= 7.524e -2
31/03/80   14.46.41

fig. 6·12 : Lung phantom , $\lambda_d = 0{\cdot}02\,\lambda_d^*$.



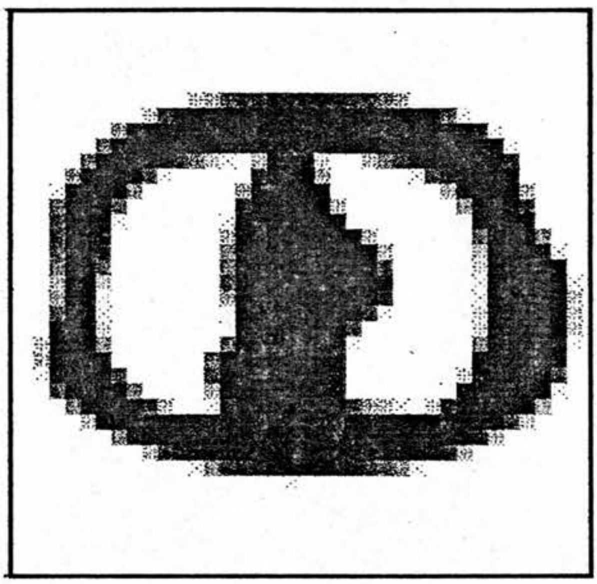
Z4503 ESTF
MIN= 0.000e-99 MAX= 7.627e -2
31/03/80 14.45.31

fig. 6·13 : Lung phantom, $\lambda_d = 0{\cdot}05\ \lambda_d^*$.



Z4504 ESTF
MIN= 0.000e-99 MAX= 7.672e -2
31/03/80 14.44.41

fig. 6·14 : Lung phantom , $\lambda_d = 0{\cdot}1\ \lambda_d^*$.



Z4505⁑ESTF
MIN= 0.000e-99   MAX= 7.811e -2
31/03/80   14.41.53

fig. 6·15 : Lung phantom , $\lambda_d = 0 \cdot 2\ \lambda_d^*$.

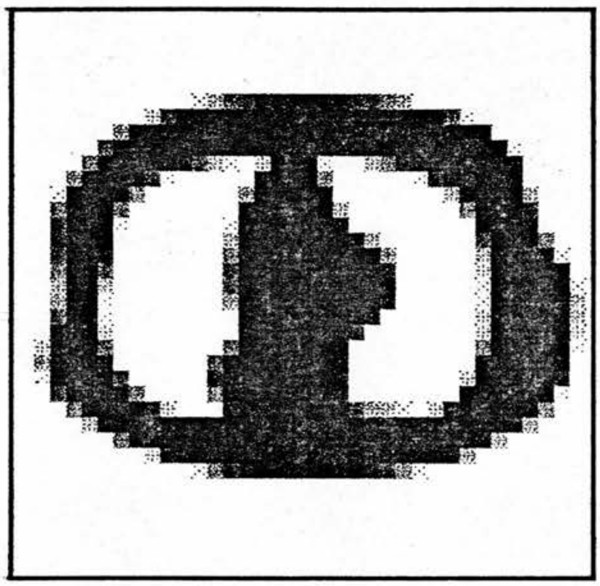
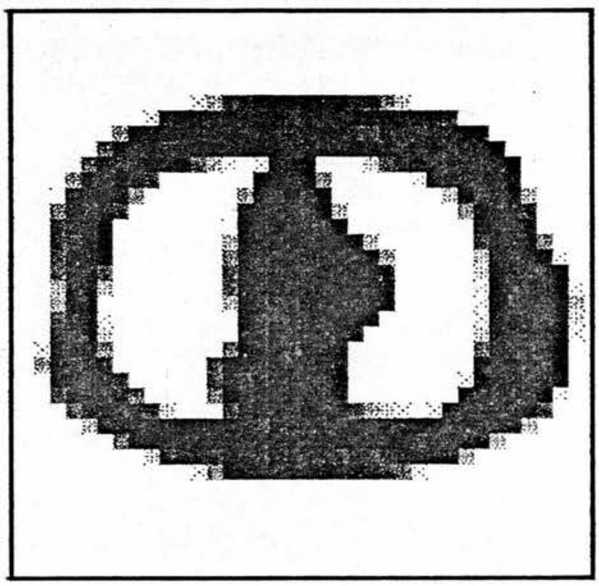


fig. 6·16 : Lung phantom, $\lambda_d = 0{\cdot}5\,\lambda_d^*$.



Z4507#ESTF

MIN= 0.000e-99   MAX= 8.119e -2

31/03/80   14.39.01

fig. 6·17 : Lung phantom, $\lambda_d = 2\,\lambda_d^*$.

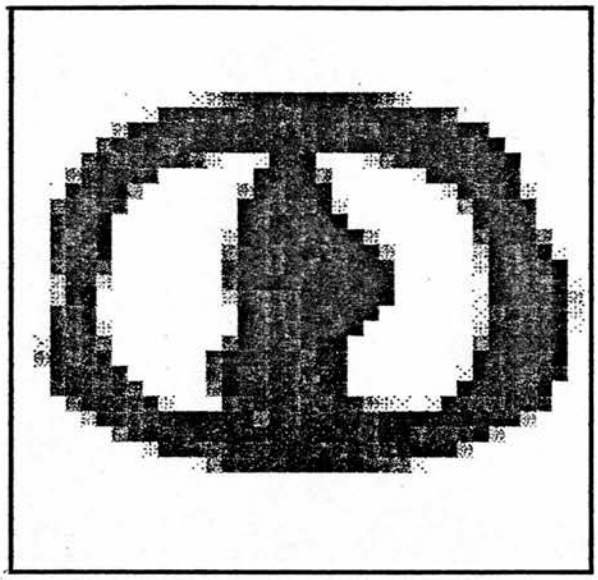
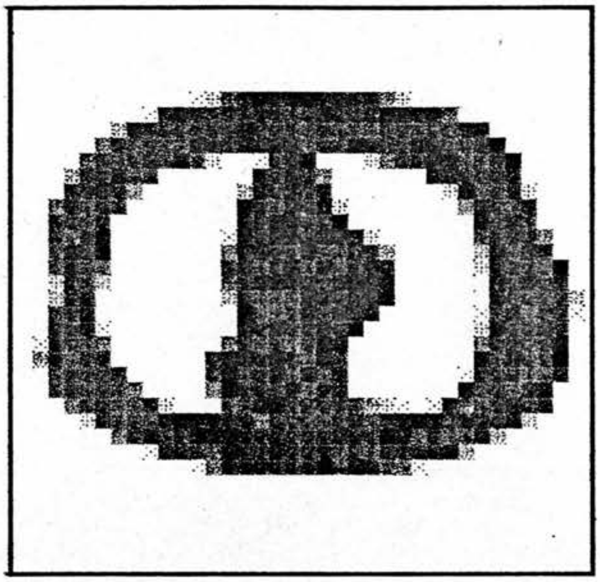
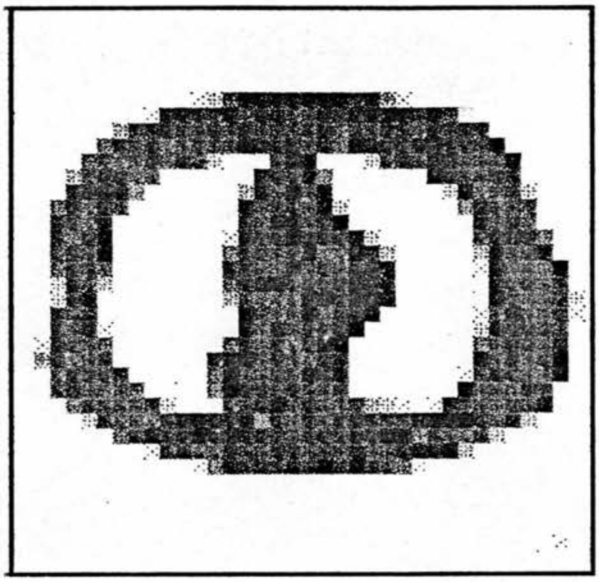


fig. 6·18 : Lung phantom, $\lambda_d = 5\lambda_d^*$.



Z4509*ESTF
MIN= 0.000e-99   MAX= 9.056e -2
31/03/80   14.36.41

fig. 6·19 : Lung phantom, $\lambda_d = 10\,\lambda_d^*$.



Z4510*ESTF
MIN= 0.000e-99 MAX= 9.494e -2
31/03/80 14.34.40

fig. 6·20 : Lung phantom, $\lambda_d = 20\,\lambda_d^*$.

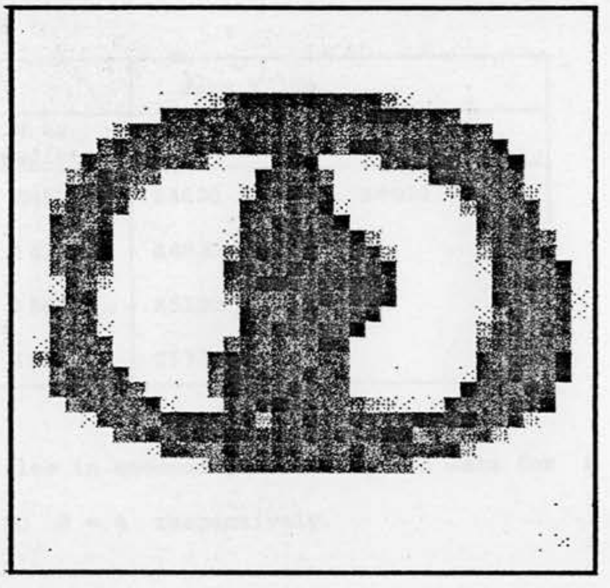


fig. 6·21 : Lung phantom, $\lambda_d = 50\,\lambda_d^*$.

·432 Scanning details

The objects and scanning conditions used for this experiment are identical with those given in §·422 except that the annulus with $\omega_{c\ rec} = 1{\cdot}000$ was not used and M was increased to 128 in all cases. Since the scanning and data processing conditions are identical (except for M) to those used on the basic data files in §·42 the data from these files may also be used in this experiment. The objects and data files are summarised below.

object	data files	
	M as predicted	other M
lung	Z4500	Z4620 - Z4623, Z4604, Z4505
disc	Z 4720	Z4820 - Z4825
annulus	Z 5020	Z5120 - Z5125
block	Z 5250	Z5320 - Z5325

In each case the files in ascending order contain data for M as predicted and M = 128 down to M = 4 respectively.

·433 Results

The resulting values of $L_{2\ rel}$ are graphed as functions of M in F·22 and the change produced in the image is illustrated for the lung in F·23 - F·27 and for the annulus in F·28 - F·32. (N.B. F·28 - F·32 are on twice the scale of F·7.)

Unfortunately, due to core restrictions on the computer, it proved impossible to process the data sets for M = 128.

From F·22 it is clear that, in terms of minimising $L_{2\ rel}$, the value of M given by §3·421 E·12 is about right. (Once again, the data for the tufnol block shows anomolous behaviour for which the author has no explanation other than the reservation expressed in section ·41.)

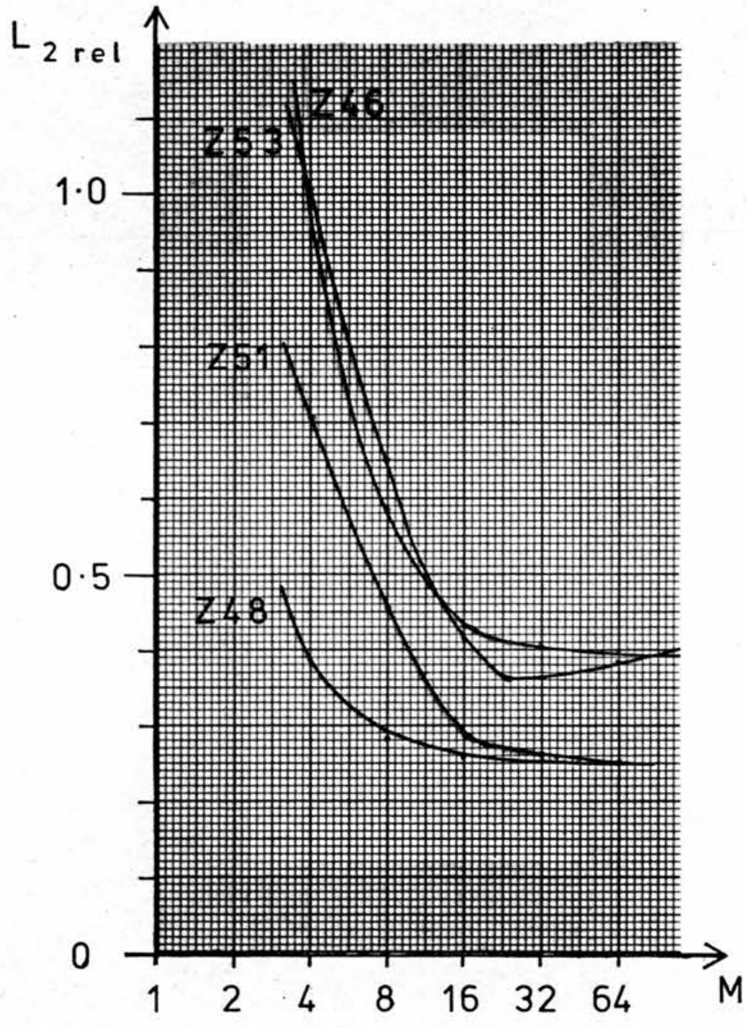
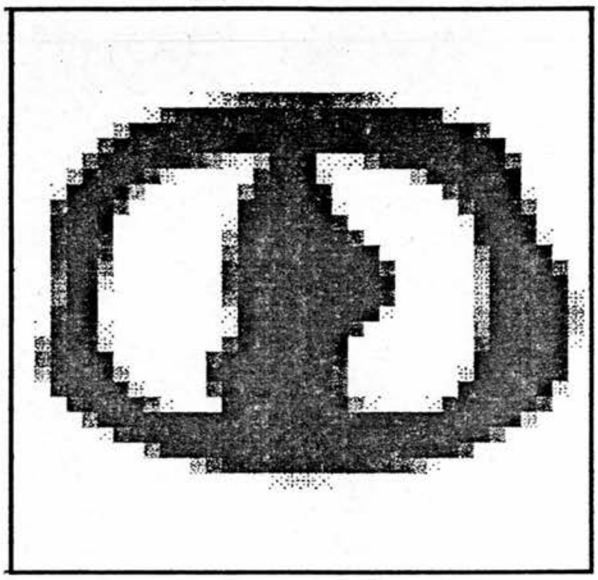


fig. 6·22 : Graph of $L_{2\,rel}(M)$.



Z4621#ESTF
MIN= 0.000@-99   MAX= 7.753@ -2
09/04/80   13.16.58

fig. 6·23 : Lung phantom, M = 64.

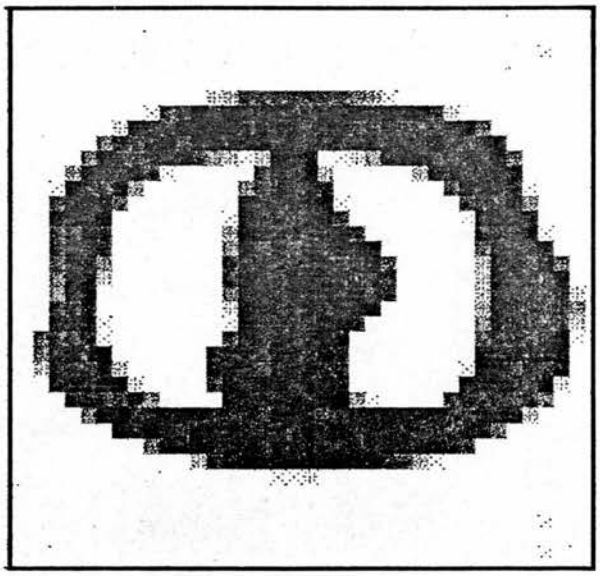
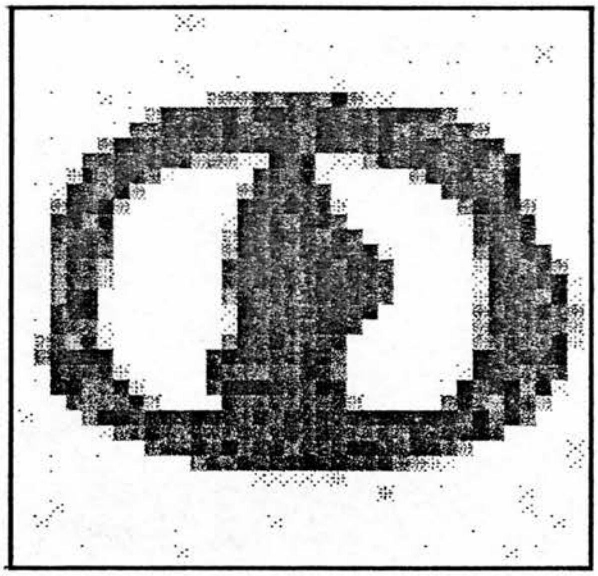
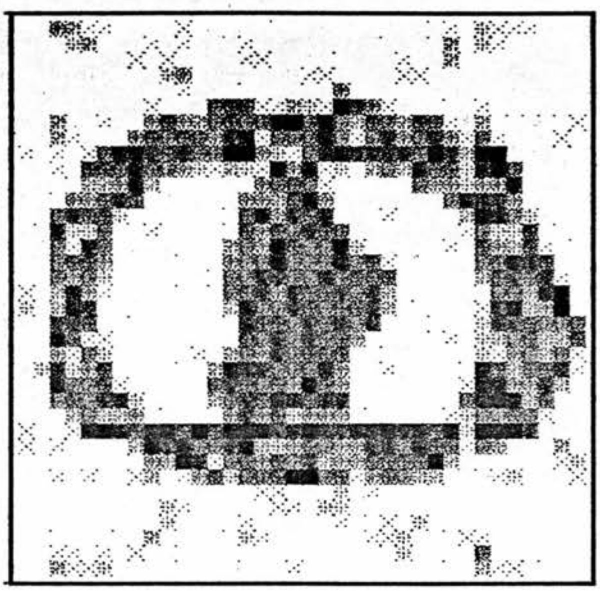


fig. 6·24 : Lung phantom, M = 32.



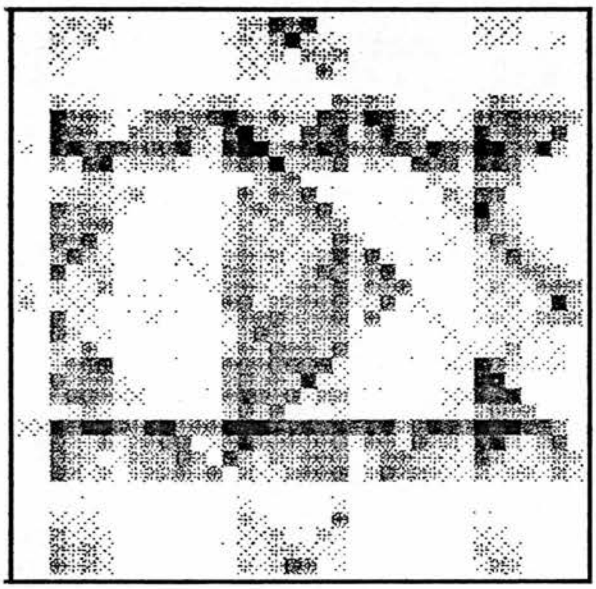
Z4623 ESTF
MIN= 0.000e-99   MAX= 8.102e -2
08/04/80   13.18.19

fig. 6·25 : Lung phantom, M = 16.



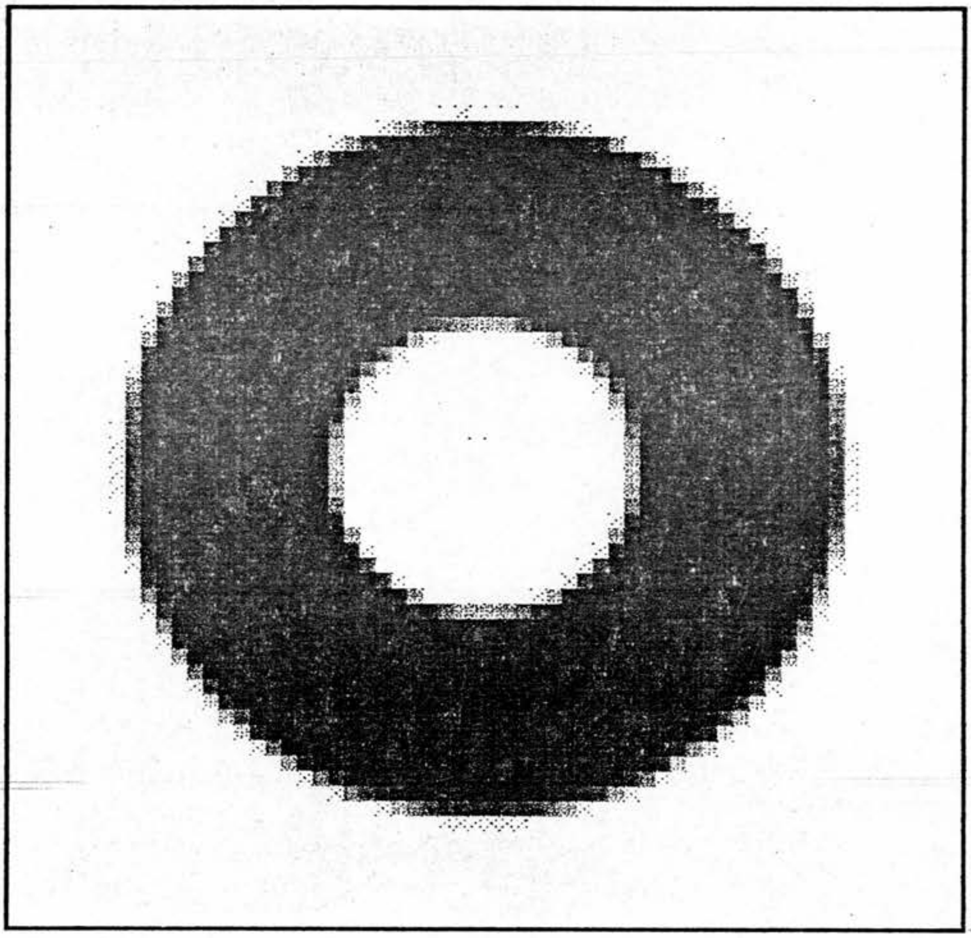
Z4604*ESTF
MIN= 0.000e-99   MAX= 1.294e -1
21/03/80   03.33.19

fig. 6·26 : Lung phantom, M = 8.



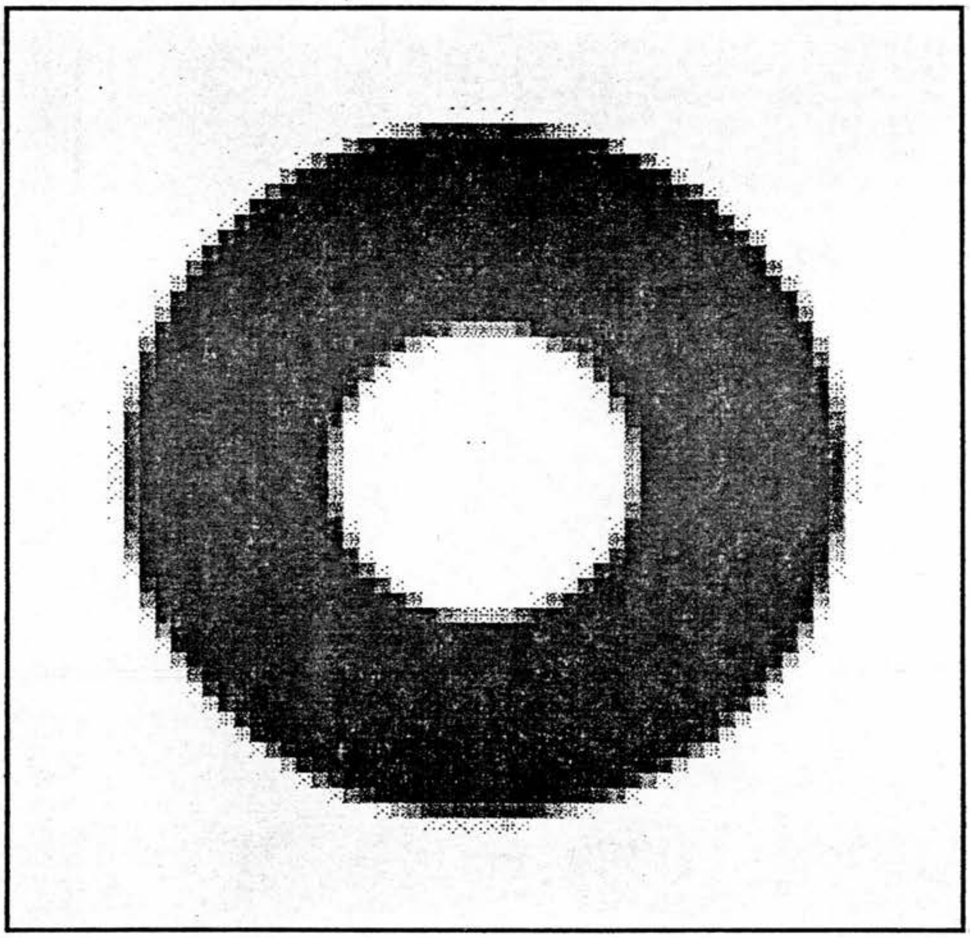
Z4605 ESTF
MIN= 0.000e-99 MAX= 2.011e -1
21/03/80 03.37.30

fig. 6·27 : Lung phantom , M = 4 .



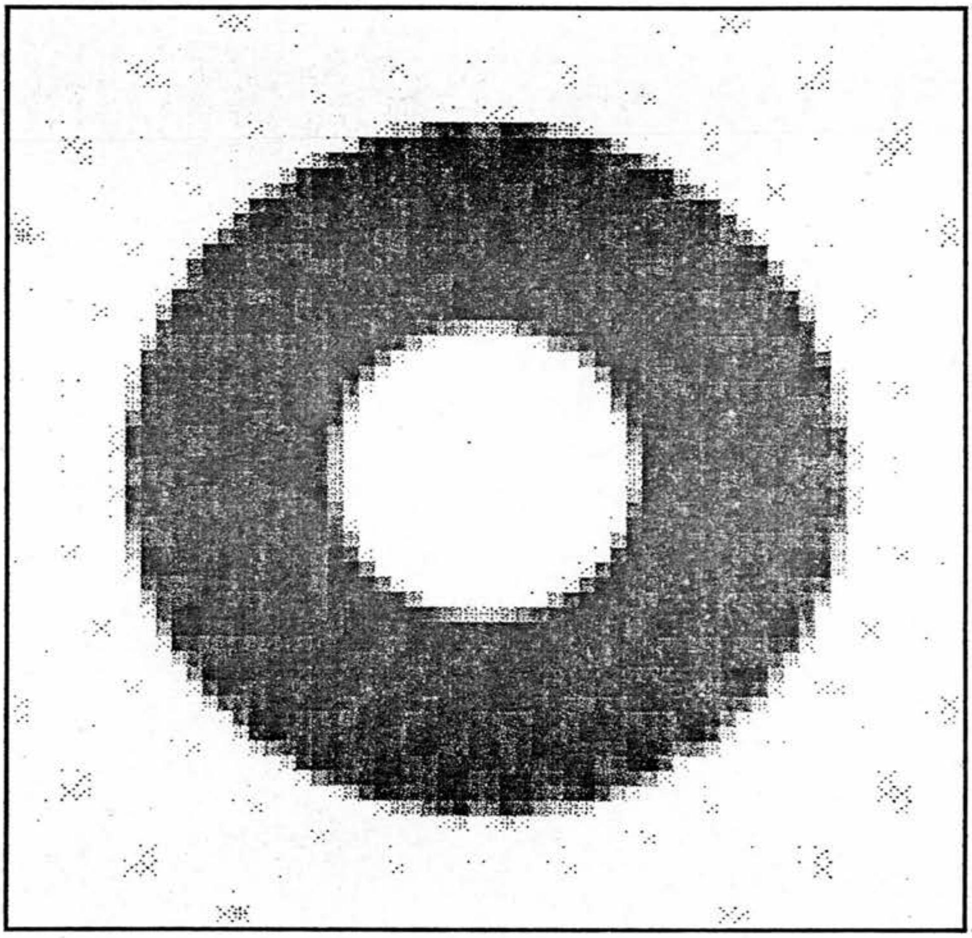
Z5121*ESTF
MIN= 0.000e-99   MAX= 8.213e -2
15/04/80   16.21.35

fig. 6·28 : Annular phantom , M = 64.



Z5122#ESTF
MIN= 0.000e-99   MAX= 8.351e -2
15/04/80   16.23.27

fig. 6·29 : Annular phantom, M = 32.



Z5123*ESTF
MIN= 0.000e-99 MAX= 9.522e -2
15/04/80 16.25.27

fig. 6·30 : Annular phantom, M = 16.

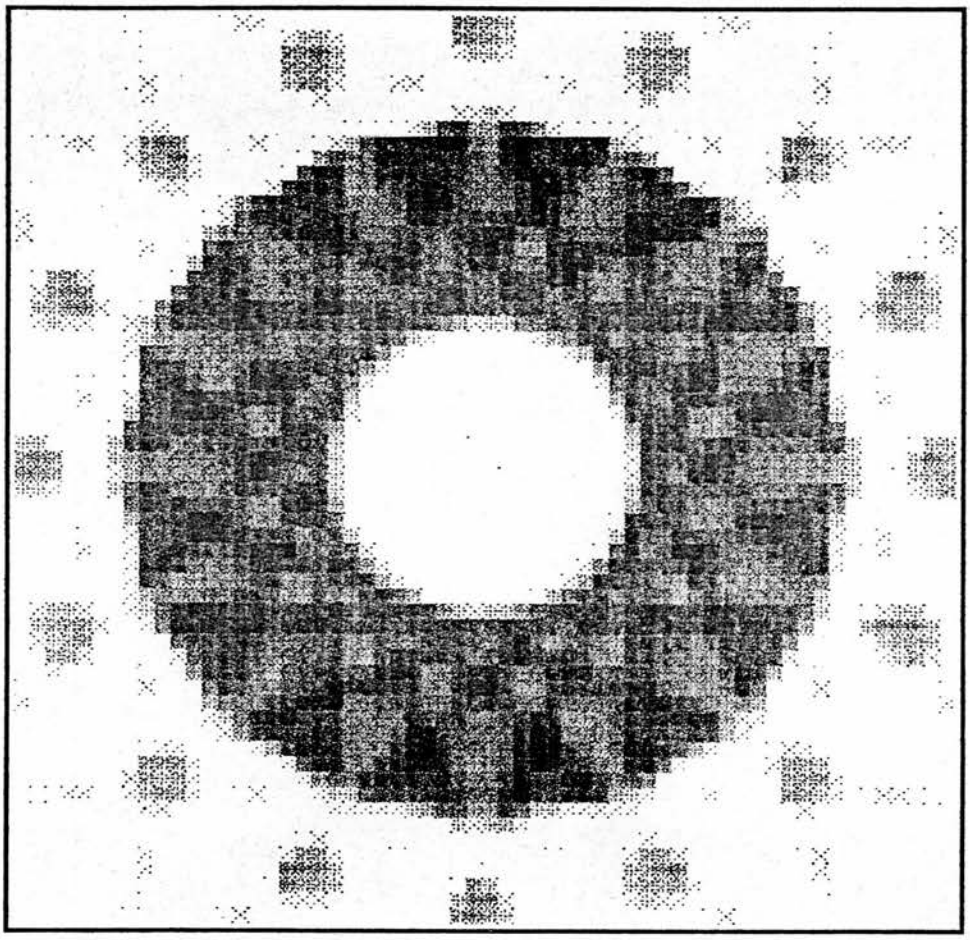
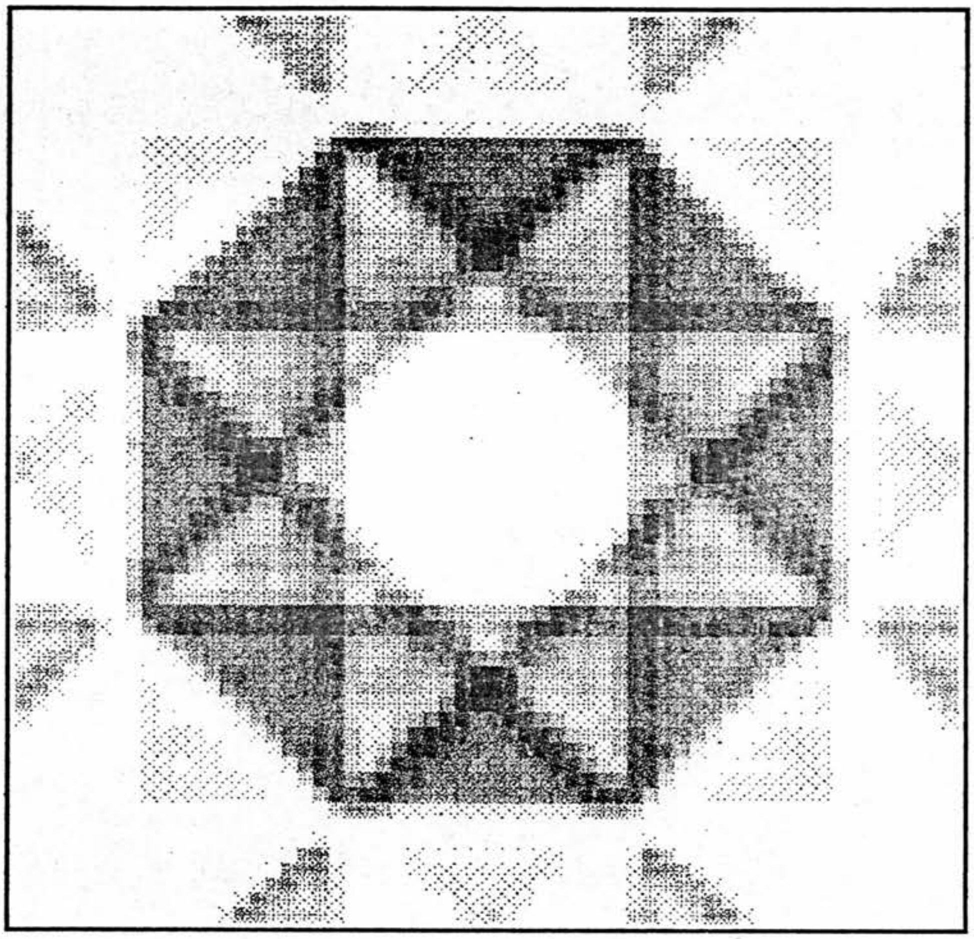


fig. 6·31 : Annular phantom, M = 8.



Z5125:ESTF
MIN= 0.000@-99   MAX= 1.553@ -1
15/04/80   16.34.28

fig.6·32 : Annular phantom, M = 4.

·44 Prediction of k

·441 Aim and method

The aim of this experiment was to test that the prediction software (in particular the criterion §3·42 E·13 and E·16) does select an appropriate value for k and hence $\omega_{c\ col}$ (assuming no deconvolution is performed).

Since k is a combination of 2a and Δr the best way to proceed would be to run PREDICT for a phantom, scan the phantom accordingly and then repeat the scans with increased and decreased 2a and Δr. This was not possible. For the reasons explained in §·31 the full range of collimaters could not be used and 2a could not therefore be varied properly. Further limitations were imposed by the software constraints of SH1 which meant that Δr could only be increased.

The method used for this experiment was to run PREDICT, scan the phantom as required and then repeat the scan but with Δr set to 2,4,8 and 16 times its original value (all other parameters, including the number of counts/cell, being kept constant).

·442 Scanning details

Only the block phantom was used for this experiment. The predicted scanning parameters used were these given in §·422. The data files used were as shown below where k^* denotes the predicted value of the normalised detector diameter and k denotes the values arising from the increased Δr.

k^*/k	data file
1	Z5250
2	Z5420
4	Z5500
8	Z5600
16	Z5720

Note the qualification concerning scan traverse length in §·41.

·443 Results

The resulting values of $L_{2\ rel}$ are graphed as a function of k^*/k in F·33 and the reconstructions illustrated in F·34 to F·38. At first sight it appears that k is predicted correctly as the curve becomes stationery at $k^*/k = 1$, however the nicely defined minimum is largely a product of the log plot. When plotted on a linear scale it is less convincing. The result is therefore inconclusive - largely due to the impossibility of reducing Δr below its original value.

·45 Prediction of ℓ

It is of interest to confirm that PREDICT does indeed estimate a value of ℓ which will give rise to desired $\omega_{c\ fil}$. When running PREDICT two sets of output were given:-

a) the predicted values for $2a, \Delta r, M, \ell$

b) the parameters in (a) but rounded to the nearest suitable scanner setting.

In practice 2a was never rounded because the software permitted only those values for which a collimater existed. The reamining parameters were usually rounded in the direction which would improve picture quality. It would therefore be surprising if there were not some variation between the value $\omega_{c\ rec}$ used in PREDICT and the value of $\omega_{c\ fil}$ realised in practise.

The values of $\omega_{c\ rec}$ and $\omega_{c\ fil}$ for different data sets are tabulated below. Values of $\omega_{c\ fil}$ at $\omega_{c\ col}$ have also been added out of general interest

object	data	$\omega_{c\ rec}$	$\omega_{c\ fil}$	$\omega_{c\ ang}$	$\omega_{c\ col}$
lung	Z4500	0·5000	0·5746	0·5117	0·5066
disc	Z 4720	0·5000	0·5746	0·5104	0·5066
annulus	Z 4950	1·000	0·8709	0·9939	0·9694
annulus	Z 5020	0·5000	0·5926	0·5104	0·5066
block	Z 5250	0·5000	0·6086	0·5101	0·5125

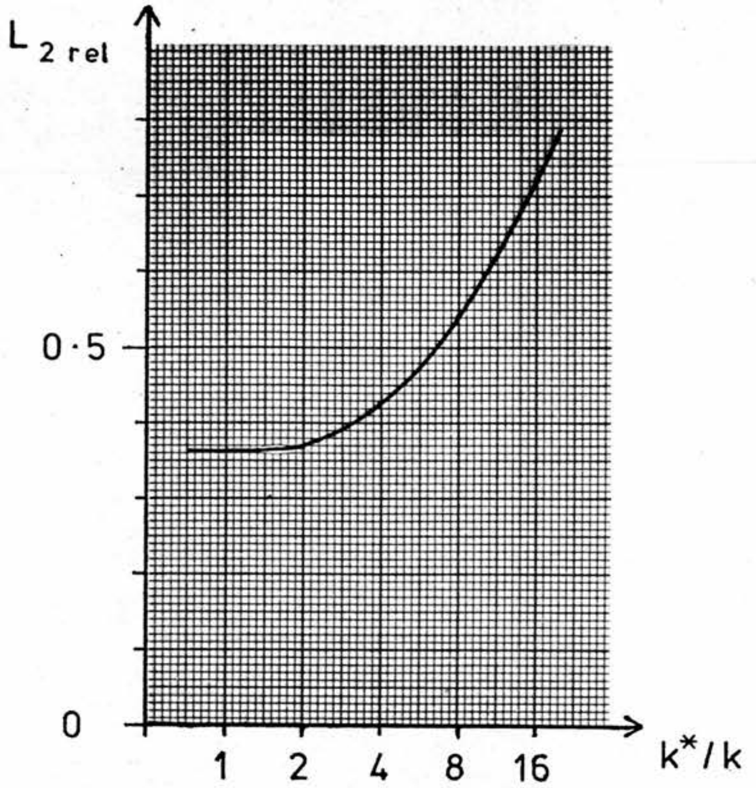


fig. 6·33 : Graph of $L_{2\,rel}(k^*/k)$.

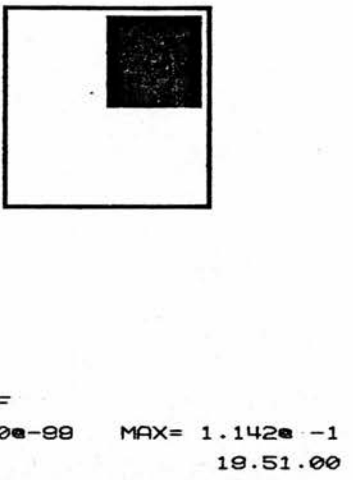


fig. 6·34 : Block phantom, $k = k^*$.

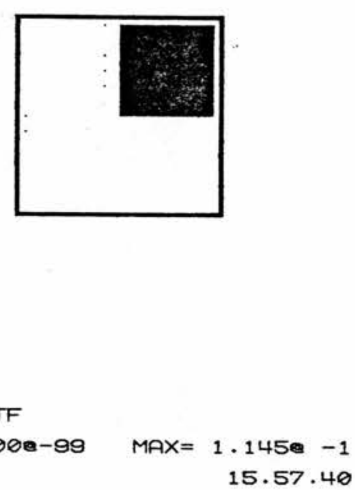


fig. 6·35 : Block phantom , $k = k^{*}/2$.

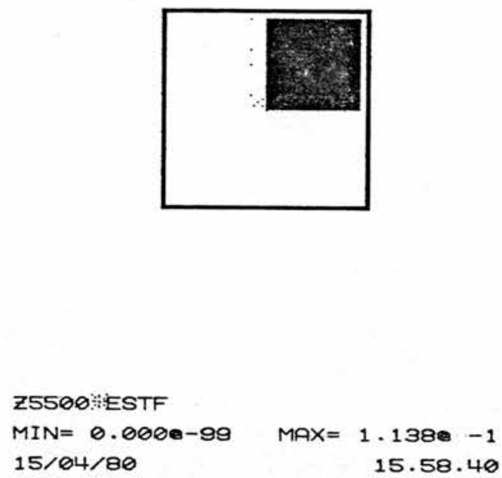


fig. 6·36 : Block phantom , $k = k^{*}/4$.

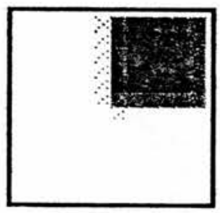
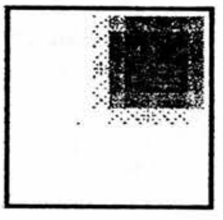


fig. 6·37 : Block phantom, $k = k^{*}/8$.



Z5720*ESTF
MIN= 0.000e-99   MAX= 1.185e -1
15/04/80   16.00.17

fig. 6·38 : Block phantom, $k = k^{*}/16$.

As is to be expected the agreement between $\omega_{c\ rec}$, $\omega_{c\ ang}$ and $\omega_{c\ col}$ is good (in fact better than 4%). The agreement between $\omega_{c\ rec}$ and $\omega_{c\ fil}$ is better than 18%, worse than one would wish though not perhaps surprising in view of the discussion on rounding of scanner parameters (see above).

·46 Effect of interpolation

To test the significance of using step function or linear interpolation in SPF all five test objects described in §·422 were back projected using both methods.

In terms of the greyscale plots no significant difference was observed. The result is illustrated for the lung phantom in F·39 (step function interpolation, file Z4540) and F·44 (linear interpolation, file Z4543). Differences for the other objects were similar.

·47 Typical edge response

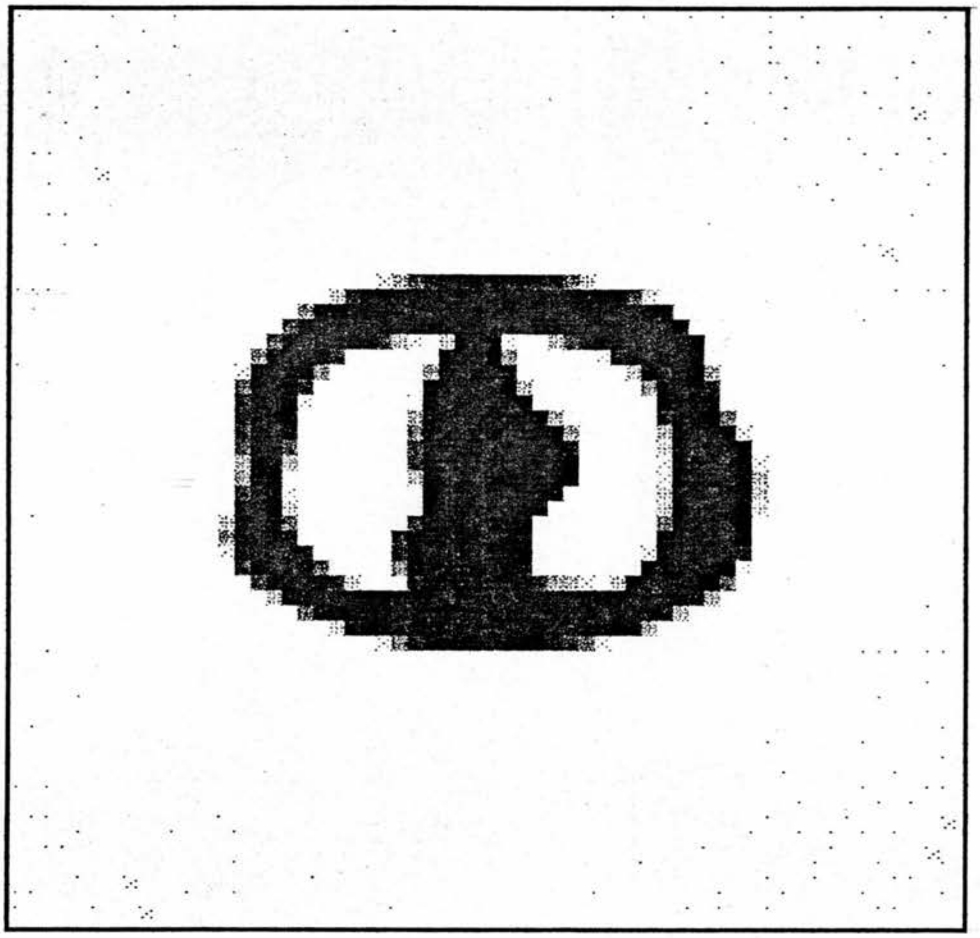
An indication is now given of the quality of edge response met in practice. The data is presented both as greyscale pictures and as graphs.

The reconstructions of the five test objects listed in §·422 are illustrated in F·39 - F·43. These reconstructions are performed without deconvolution and with step function interpolation in SPF. The pictures show ESTF after removal of negative values and should be compared with F·5 - F·8 which show the original test objects.

Graphical sections through this data (taken as described in §·41(c)) are given in F·45 - F·53, in this case negative values have not been removed.

·48 Effect of deconvolution

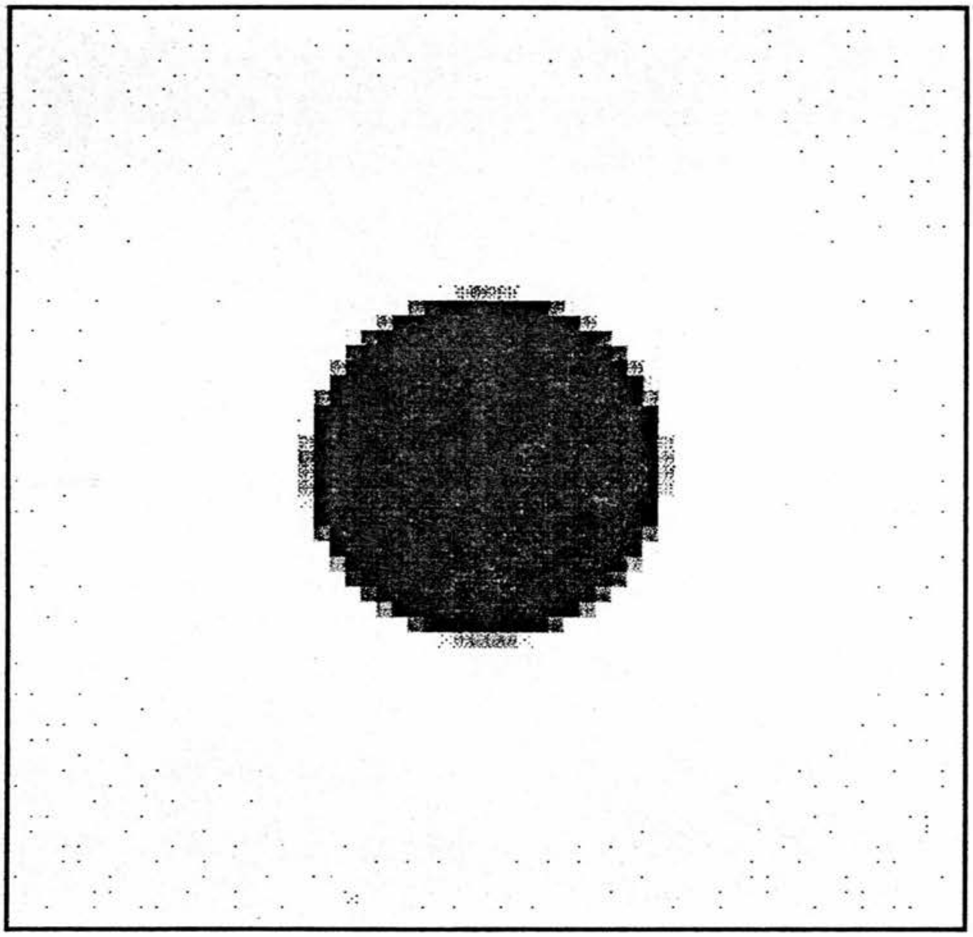
In order to access the effect of deconvolution it was decided to reprocess the data for the give test data sets of §·422 but using a full run of GQ. Unfortunately it proved impossible to complete the processing of the



Z4540*ESTF
MIN= 0.000e-99 MAX= 7.897e -2
01/05/80 22.55.30

(no deconv. & step function interp.)

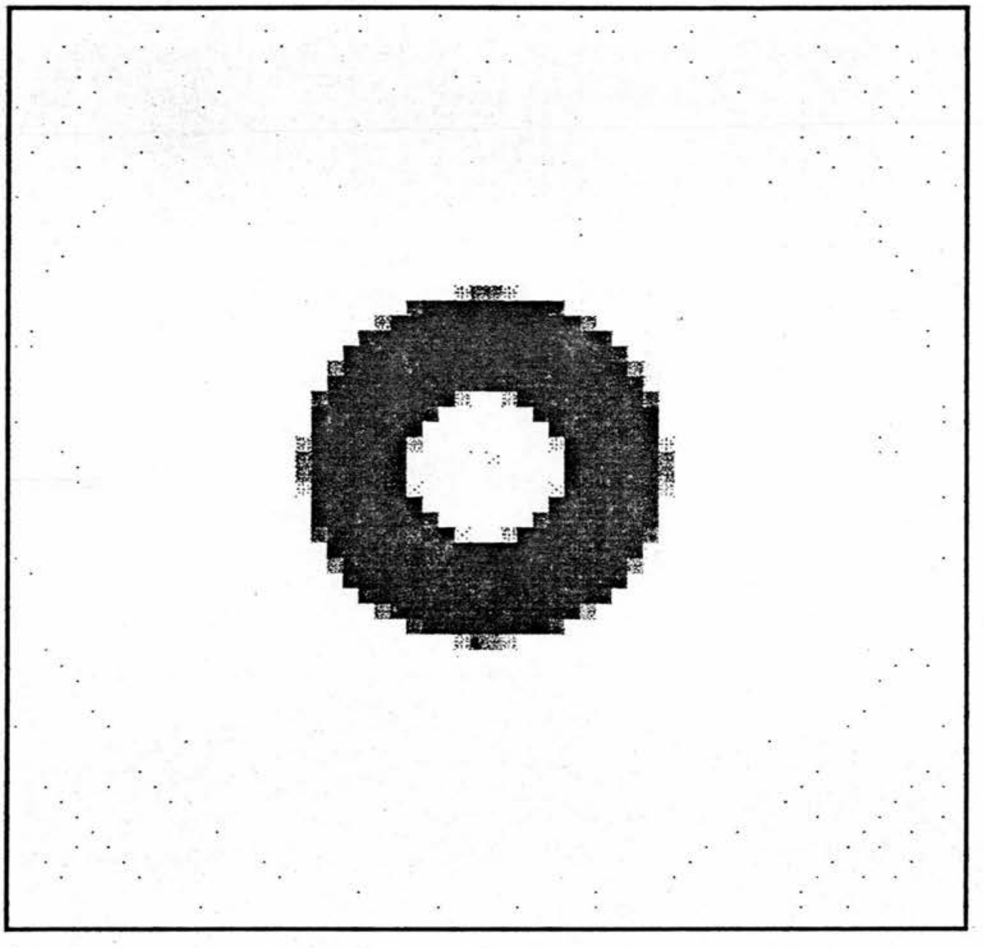
fig. 6·39 : Reconstruction of lung phantom.



Z4740*ESTF
MIN= 0.000e-99 MAX= 7.848e -2
01/05/80 22.58.59

(no deconv. & step function interp.)

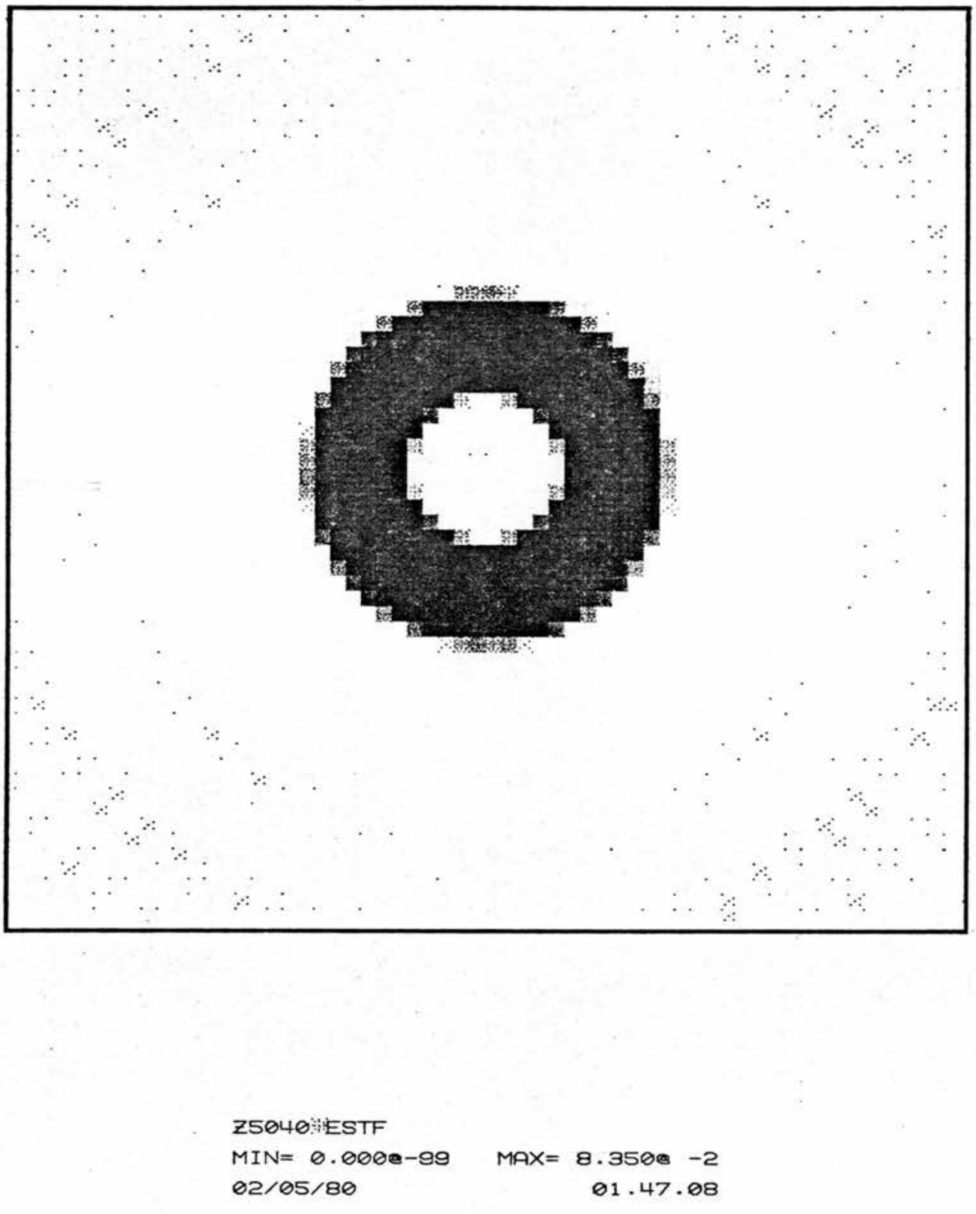
fig. 6·40 : Reconstruction of disc phantom .



Z4940 ESTF
MIN= 0.000e-99 MAX= 8.747e -2
01/05/80 23.02.29

(no deconv. & step function interp., $\omega_{c\ rec} = 1 \cdot 0$)

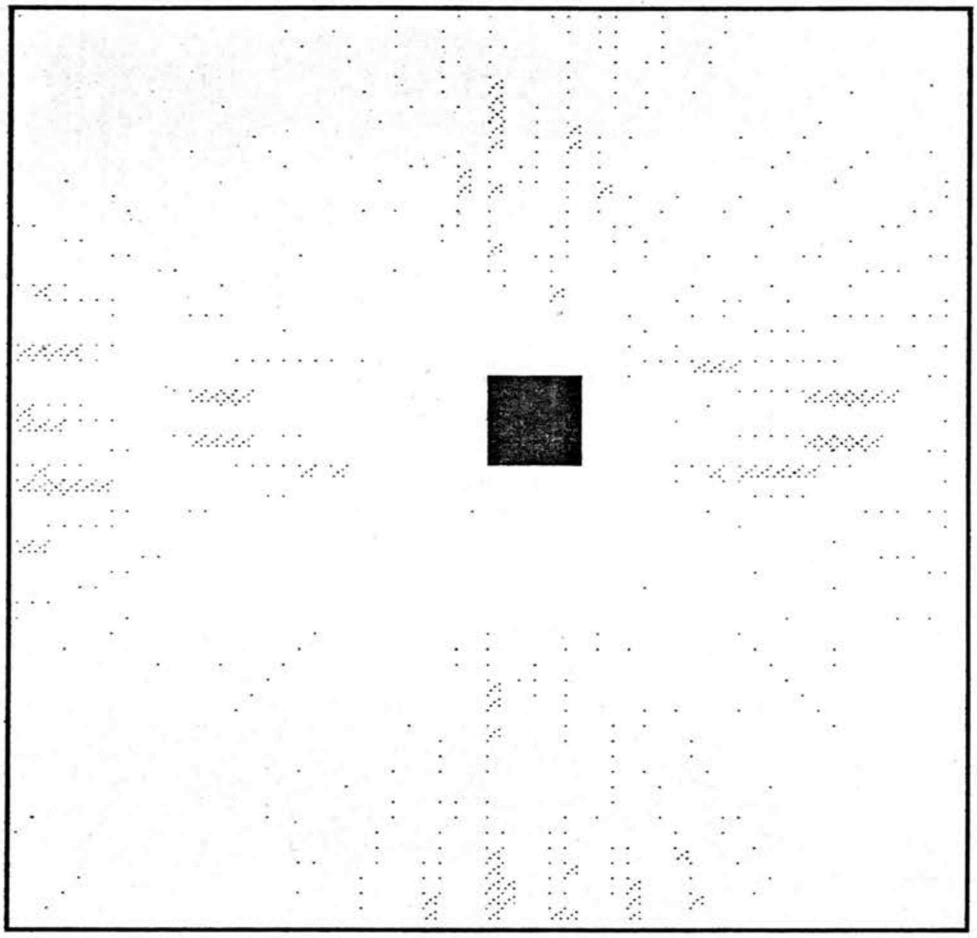
fig. 6·41 : Reconstruction of annular phantom .



Z5040*ESTF
MIN= 0.000e-99 MAX= 8.350e -2
02/05/80 01.47.08

(no deconv. & step function interp., $\omega_{c\ rec} = 0\cdot5$)

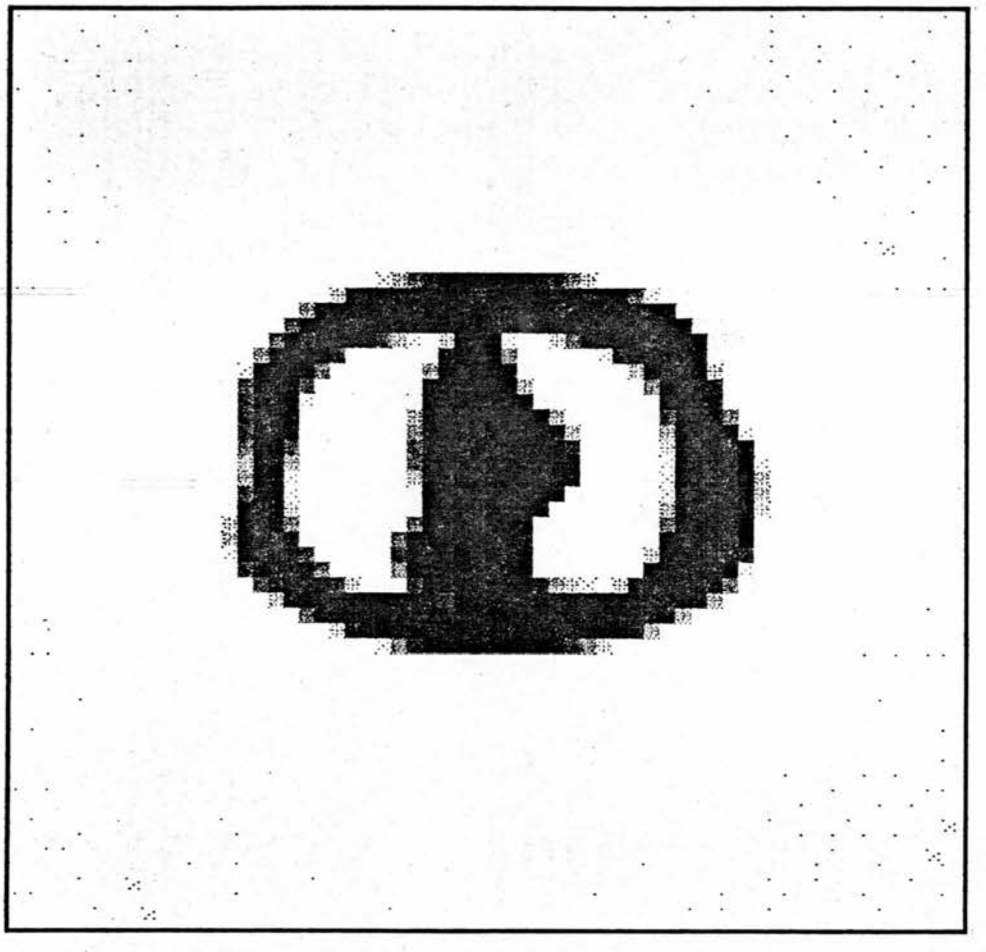
fig. 6·42 : Reconstruction of annular phantom.



Z5240*ESTF
MIN= 0.000e-99   MAX= 1.186e -1
02/05/80   01.59.41

(no deconv. & step function interp.)

fig. 6·43 : Reconstruction of block phantom.



Z4543*ESTF
MIN= 0.000e-99   MAX= 7.951e -2
01/05/80   22.57.59

(no deconv. & linear interp.)

fig. 6·44 : Reconstruction of lung phantom.

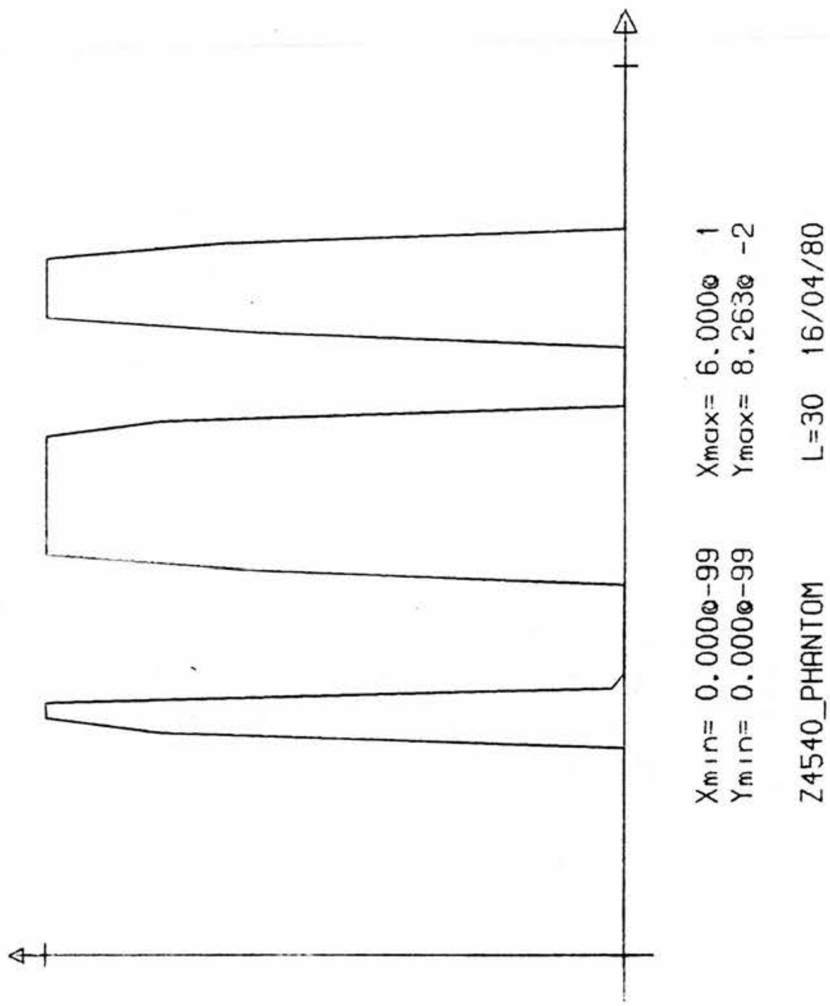


fig.6·45 : Lung phantom section.

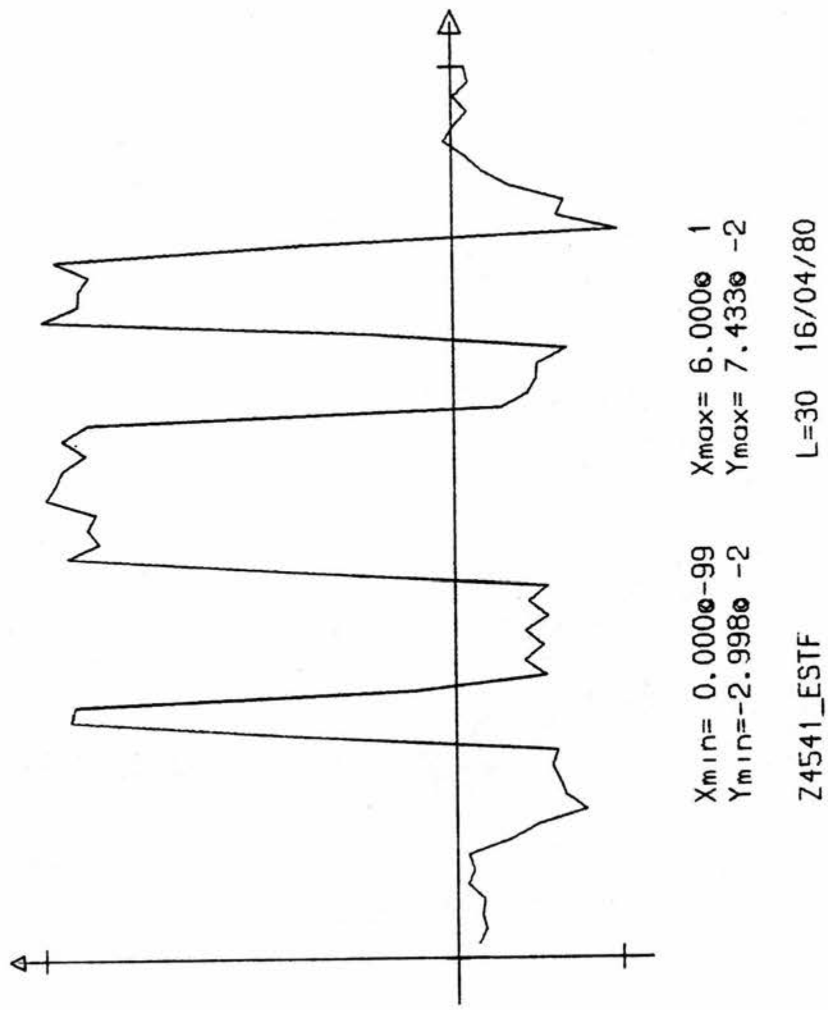


fig. 6·46 : Lung reconstruction section.

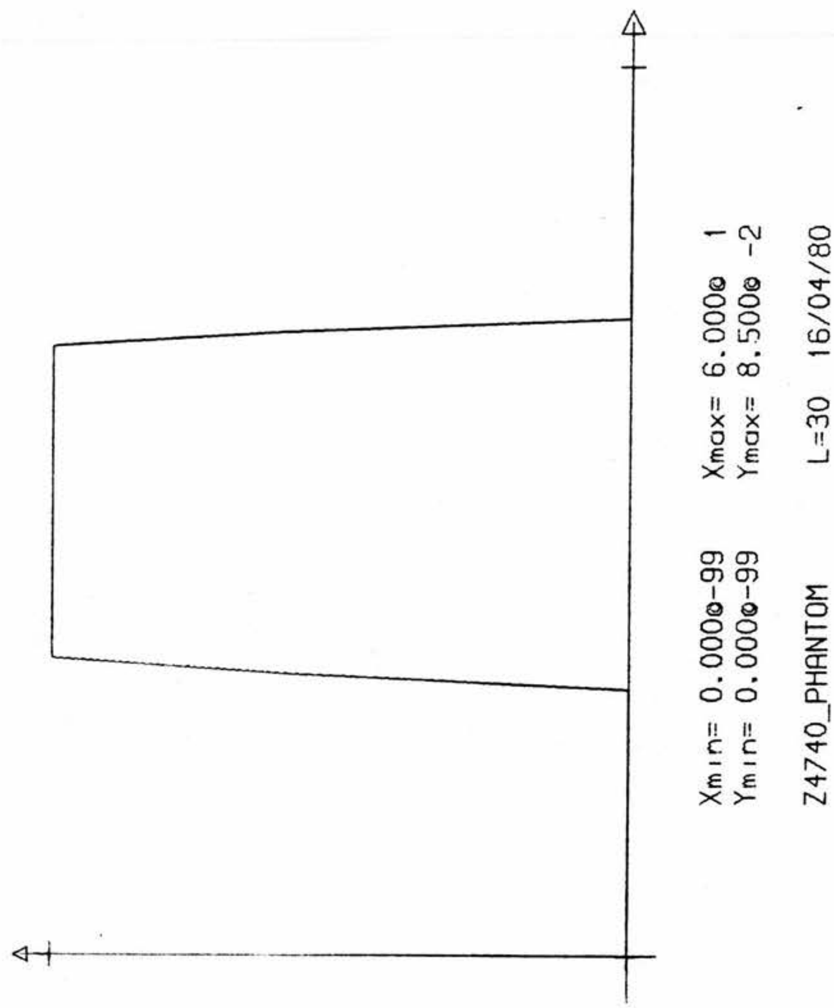


fig. 6·47 : Disc phantom section.

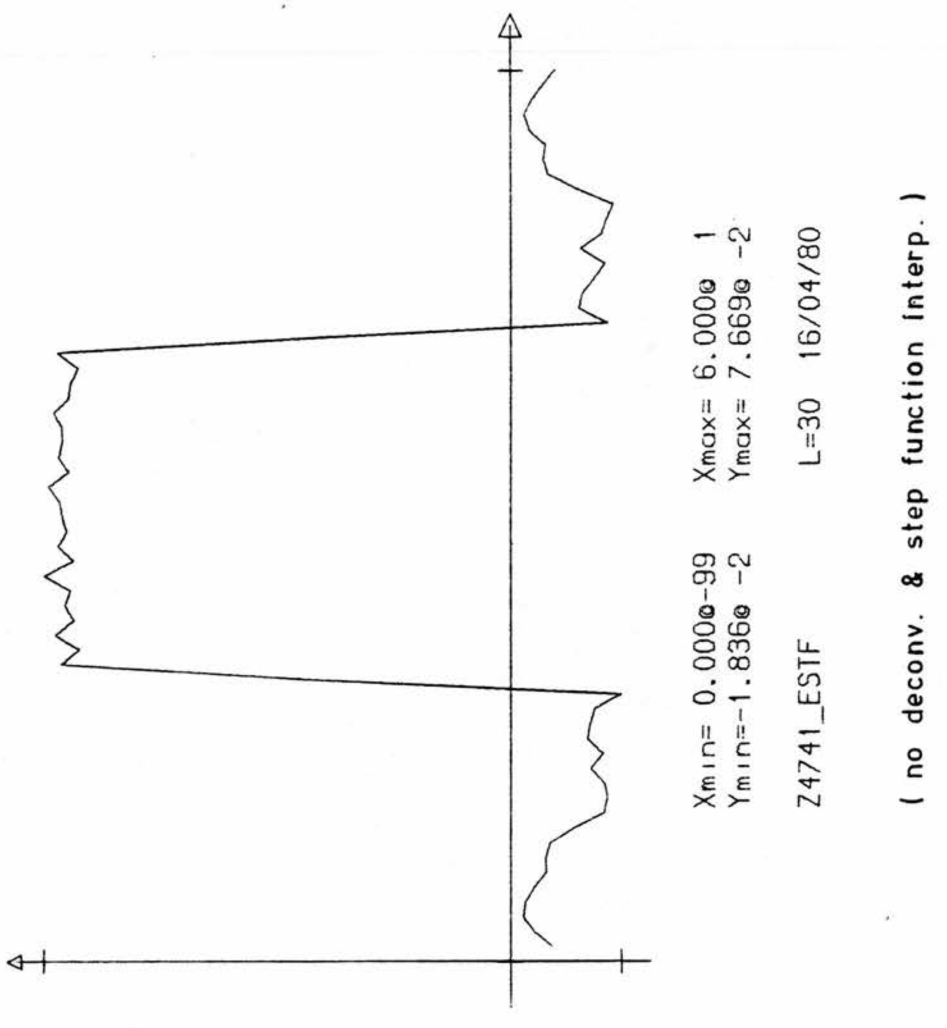


fig. 6·48 : Disc reconstruction section.

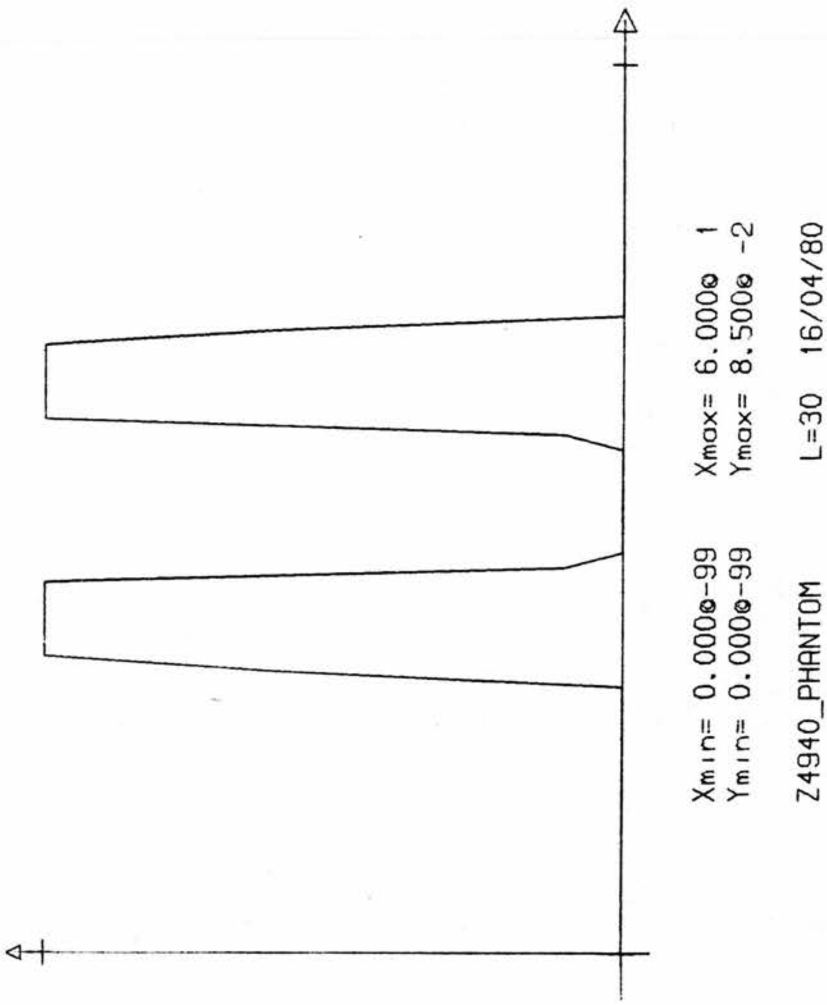


fig. 6·49 : Annular phantom section.

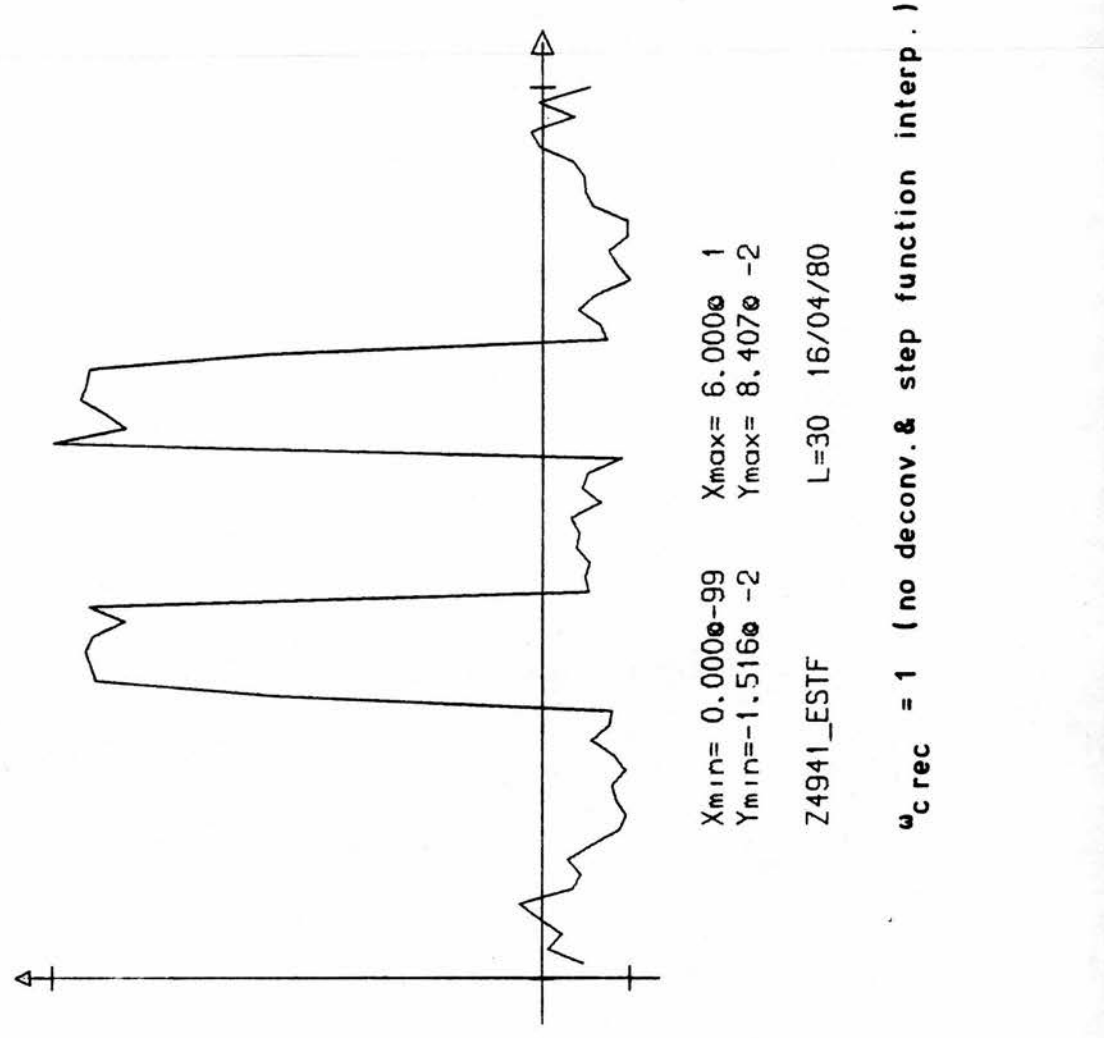


fig. 6·50 : Annular reconstruction section.

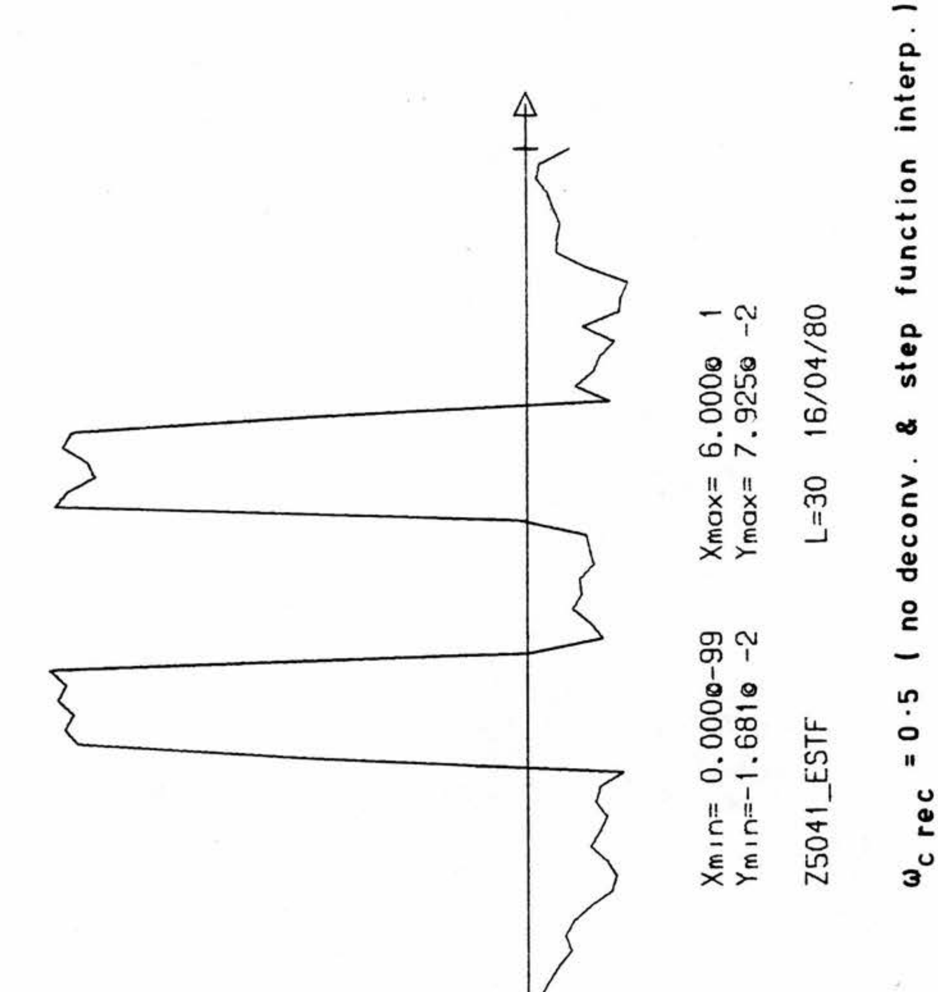


fig. 6·51 : Annular reconstruction section.

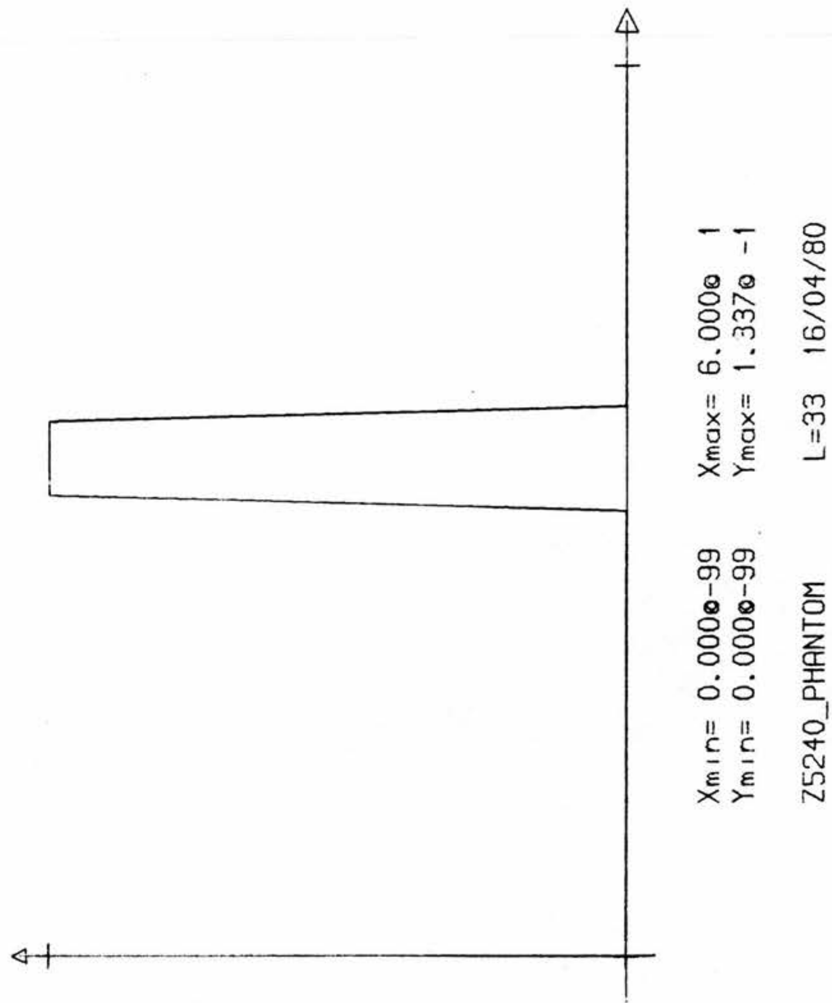


fig. 6·52 : Block phantom section.

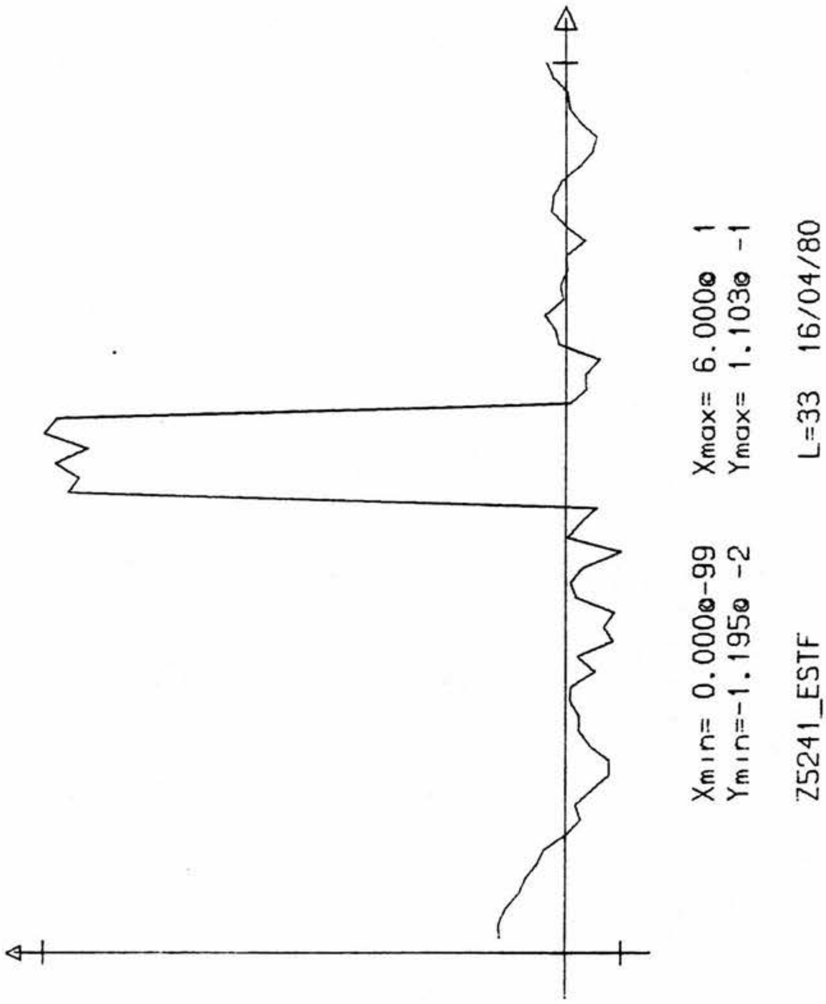


fig. 6·53 : Block reconstruction section.

lung data (file Z4500) or the annulus data (with $\omega_{c\ rec} = 1{\cdot}0$) (file Z4950) due to space constraints on the EMAS (see §5·31) computing system.

The effect of deconvolution on the other data sets may be seen in F·54 - F·56 which show the disc, the annulus with $\omega_{c\ rec} = 0{\cdot}5$ and the block respectively. These should be compared to the reconstruction without deconvolution (F·40, F·42 and F·43) and to the original phantoms (F·6, F·7 and F·8). The results may be summarised as:-

disc : deconvolution has caused a significant deterioration due to over smoothing of the picture.

annulus : deconvolution has made little difference within the central area $|r| < D/2$.

block : deconvolution has caused some worsening of the artifacts within the area $|r| < D/2$.

The effect on the cut-off frequencies is summarised in the table below:-

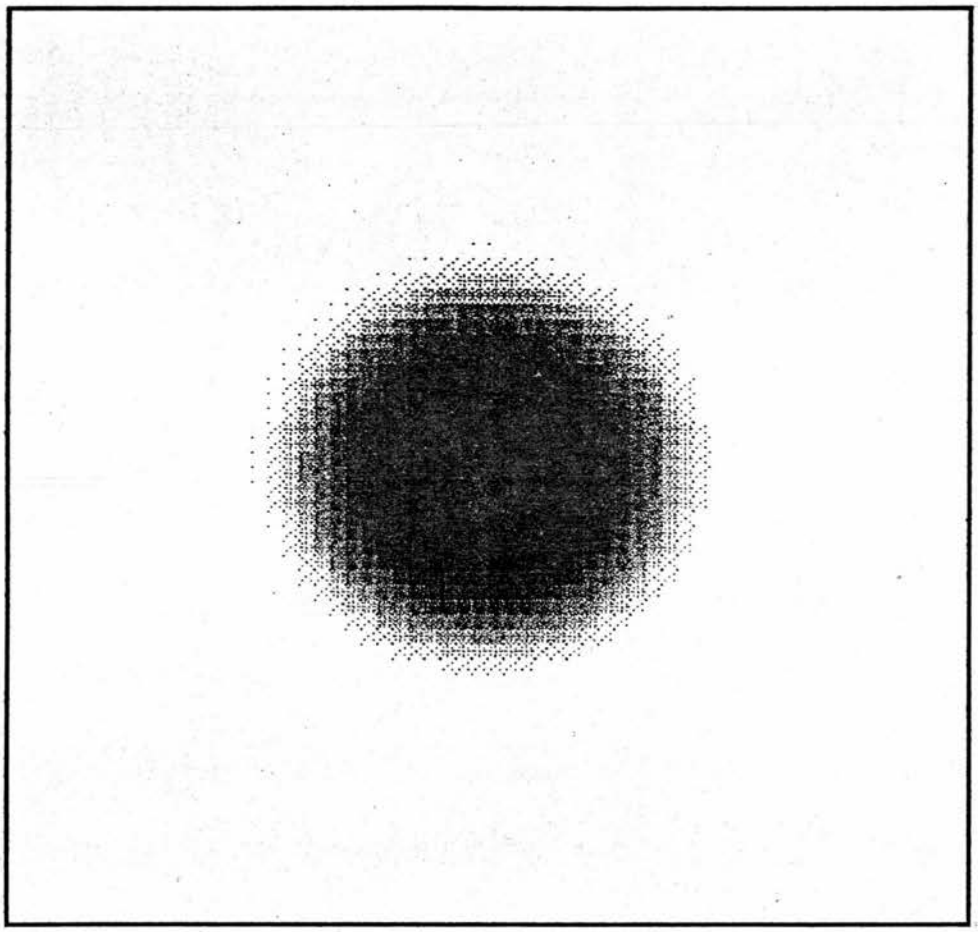
object	no deconvolution			with deconvolution		
	file	$\omega_{c\ fil}$	$\omega_{c\ col}$	file	$\omega_{c\ fil}$	$\omega_{c\ col}$
disc	Z4740	0·5746	0·5066	Z4745	0·06925	0·8973
annulus	Z5040	0·5926	0·5066	Z5045	0·6231	0·7493
block	Z5240	0·6086	0·5125	Z5245	0·9646	0·4479

The author finds the value of $\omega_{c\ fil}$ for the disc after deconvolution extremely surprising but is unaware of anything that would explain it.

·5 Discussion

In this chapter the aim has been to present data which will act as a check on the performance of the software. Within the limitations of the test scanner the conclusions are:-

1) the criterion §2·532 E·7 selects a value for $\hat{f}$ which not only minimises $\|\nabla^2 f\|_{B^2}$ but also comes close to minimising $L_{2\ rel}$ (which is closely related to $\|f - \hat{f}\|_{B^2}$).



Z4745#ESTF
MIN= 0.000e-99   MAX= 8.848e -2
01/05/80   01.41.04

fig. 6·54 : Disc reconstruction with deconvolution.

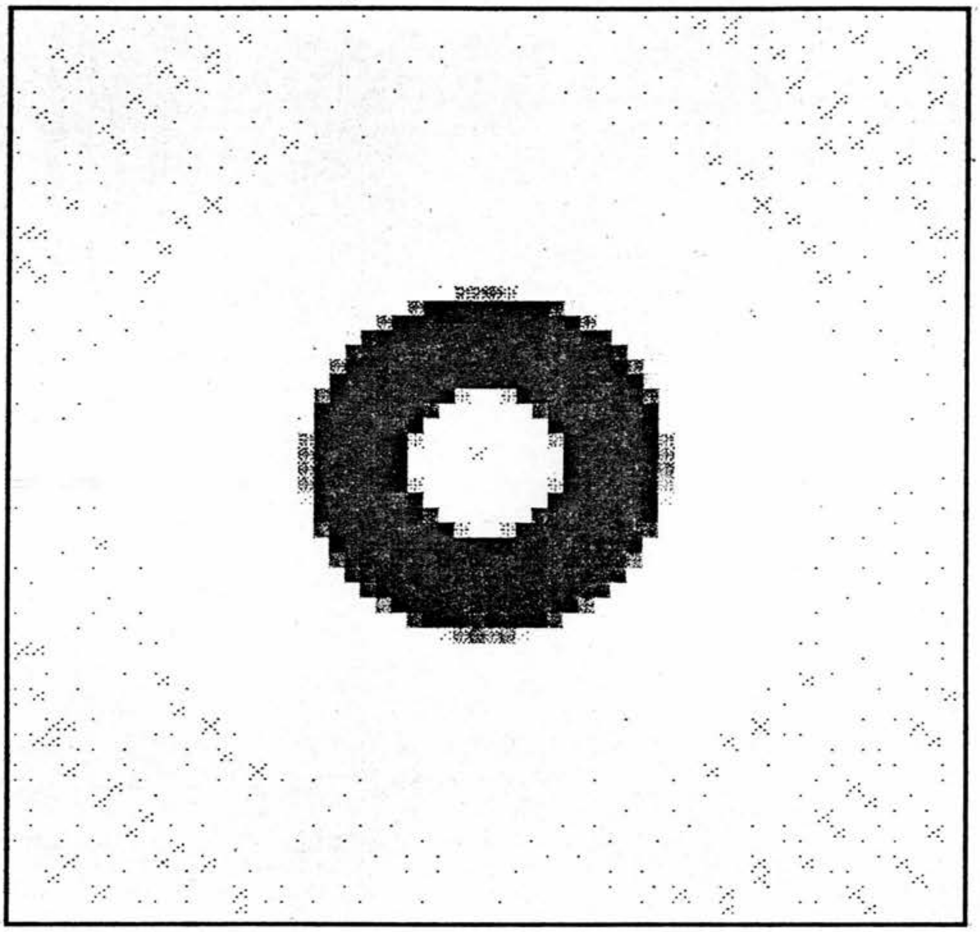
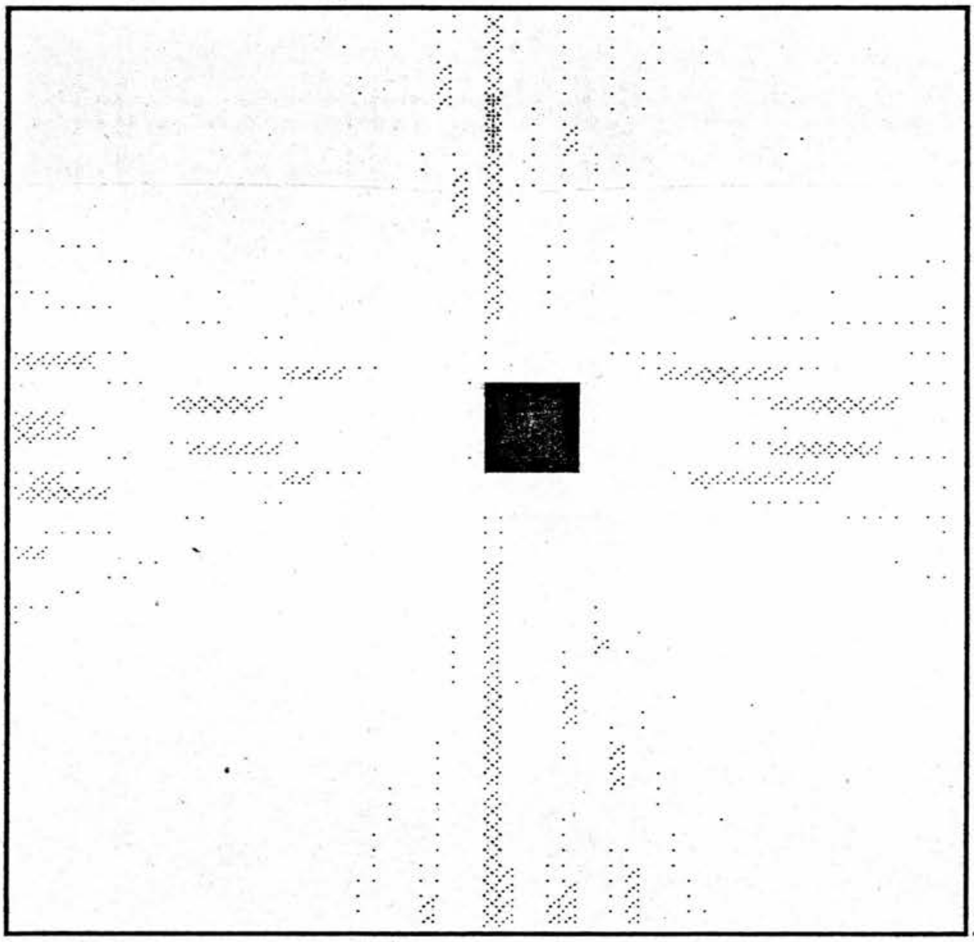


fig. 6·55 : Annular reconstruction with deconvolution.



Z5245*ESTF
MIN= 0.000e-99 MAX= 1.350e -1
02/05/80 02.31.27

fig. 6·56 : Block reconstruction with deconvolution.

2) the PREDICT software selects a value of M which is approximately optimal in terms of minimising $L_{2\ rel}$.

3) the data collected on the value of k is consistent with PREDICT selecting the optimal value but does not prove it.

4) the PREDICT software selects a value for ℓ so that the calculated value of $\omega_{c\ fil}$ is within about 18% of the required $\omega_{c\ rec}$. If one takes into account the reservations about the block data (see §·41) and the fact that none of the objects are scanned precisely as predicted (see §·45) then the agreement is likely to be better than this.

5) for the image resolution and display resolution considered here, the difference between step function and linear interpolation is negligible.

6) on the basis of the three reconstructions used deconvolution does not seem to be of any significance for the resolutions being considered.

Chapter 7: Concluding Discussion

·1 Work covered in thesis

If we now pause to look back over the work given in the thesis it will be seen to contain the following.

In chapter 1 a brief overview of some of the extant literature was given and in particular various shortcomings of the reconstruction techniques were discussed. In particular it was shown that $\mathcal{R}^{-1}$ is discontinuous and that many of the reconstruction methods coped with the resulting instability in rather ill defined ways.

In chapter 2 this led to a more careful definition of a model for transmission scanning with an isotope source and consideration of this model yielded a more complex reconstruction algorithm which includes techniques for dealing with $\mathcal{R}^{-1}$ in a more controlled manner.

In chapter 3 consideration was given to relating the patient parameters to the scanner parameters, a process which is fundamental to a proper machine design. In particular it was shown that the scanner parameters could all be reduced to functions of the section and the constant $\omega_{c\ rec}$ (and one or two other variables) and this relationship was reduced to workable software. In addition the relation between the patient parameters and $\omega_{c\ rec}$ was also indicated though not in great detail. Combining these relations it is possible to choose a patient section and find the $\omega_{c\ rec}$ corresponding to an acceptable reconstruction quality and then use this $\omega_{c\ rec}$ to predict the required scanner parameters.

Because of the length and complexity of the ideas in chapters 2 and 3 it was felt that some check on the accuracy of the ideas should be made. To this end a small test scanner was built (see chapter 4) and most of the results of chapters 2 and 3 reduced to computer software (chapter 5).

In chapter 6 data was presented which represents the effect of combining the maths of chapters 2 and 3 with the equipment of chapter 4 and the software of chapter 5. The author's overall interpretation of the results of chapter 5 is that they show up no fundamental flaws in the mathematics and in general confirm its accuracy (see §·2 however).

·2 Work not covered in thesis

There are at least four items which should have been considered in this thesis and have been omitted; zero correction, relation between $\omega_{c\ rec}$ and patient parameters, word length required in data processing and mechanical accuracy required in scanner construction.

It has been pointed out that as a result of the fact that $(\underline{S})_0 = 0$, the steady state component of $\hat{\underline{p}}$ will always be removed, i.e. that

$$\underline{\underline{s}}(\hat{\underline{p}} + c\ \underline{u}) = \underline{\underline{s}}\ \hat{\underline{p}}$$

where $\underline{u} = (1,1,1,\ldots,1)'$ and c is any constant. There is thus a whole family of projections $\hat{\underline{p}}(\phi_j)$ which will give rise to the same reconstruction. For this reason, it has been said that it is impossible to obtain accurate attenuation coefficients from reconstructions. The author had originally considered this conclusion to be incorrect and proceeded accordingly. More careful consideration has in fact led the author to conclude that there is some truth in the matter but that the problem can be satisfactorily dealt with by adding a small correction factor to $\underline{\underline{s}}$ (in fact a correction required as a result of the zero of S at $R = 0$). Unfortunately lack of time has made it impossible to add the necessary alterations either to the results of chapters 2 and 3 or to the software of chapter 5.

The relationship between $\omega_{c\ rec}$ and the patient parameters was indicated in §3·5 and while enough detail has been given for the interested reader to develop it for himself it would have been preferable to include it in chapter 3.

It had been hoped to establish the word length required for data processing and the design tolerances required in the scanner construction - both highly relevant to anyone contemplating building a scanner.

None of these last three points were taken any further due to lack of time.

·3 Closing discussion

The overall aim of this thesis has been to derive a model for the scanning process and to consider its implications in sufficient detail to obtain information on how to design the hardware and software of a transmission scanner. This has led to work stretching from the functional analysis of to the building of a test scanner and, not surprisingly, covers a wide range of subject matter in between. Such an attempt to cover the whole subject of transmission scanning as opposed to the widespread practice of considering just small parts in isolation does seem to have advantages. For example most of the work required for the prediction theory of chapter 3 originated, unintentionally, in §2·54 while considering numerical errors in evaluating $s * p$ on a computer.

It is the author's hope that perhaps this idea of thinking in terms of a coherent overall theory may become a general trend in the literature until such time as a complete theory is developed on how best to design and build all parts (hardware and software) of a transmission scanner.

Appendix 1·1 : Definitions and Theorems

·1 Definitions

·11 Continuous Convolution

Given h, q defined on $\mathbb{R}^n$ their continuous convolution $h * q$ is

$$(h * q)(\underline{x}) = \int_{-\infty}^{\infty} h(\underline{x} - \underline{\xi}) q(\underline{\xi}) d\underline{\xi}$$

·12 Continuous Fourier Transform

If $\underline{x} \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{C}$ then the function $F : \mathbb{R}^n \to \mathbb{C}$ defined by

$$F(\underline{X}) = \int_{-\infty}^{\infty} f(\underline{x}) e^{-2\pi i < \underline{x}, \underline{X} >} d\underline{x}$$

(where $< \underline{x} \cdot \underline{X} >$ denotes an inner product) is called the continuous Fourier transform (C.F.T.) of f. The inverse relationship is

$$f(\underline{x}) = \int_{\infty}^{\infty} F(\underline{X})\ e^{2\pi i < \underline{X}, \underline{x} >}\ d\underline{X}$$

These two mappings are referred to as the forward and inverse or reverse transforms respectively. When useful, the mappings will be thought of as operating on function spaces and denoted by

$$F = \mathcal{F}_n f \quad \text{and} \quad f = \mathcal{F}_n^{-1} F.$$

Note the obvious simplifications.

$$\text{if} \quad n = 1 \quad < \underline{x}, \underline{X} > \; = xX$$

$$\begin{aligned} \text{if} \quad n = 2 \quad < \underline{x}, \underline{X} > \; &= x_1 X_1 + x_2 X_2 \quad \text{where } \underline{x} = (x_1, x_2) \text{ and } \underline{X} = (X_1, X_2) \\ &= r\,R \cos(\theta - \phi) \text{ where } \underline{x} = (r \cos \theta,\ r \sin \theta) \text{ and} \\ &\qquad \underline{X} = (R \cos \phi,\ R \sin \phi) \end{aligned}$$

·13 Discrete convolution

Given two sequences h_j, q_j $j = -\infty(1)\infty$ their discrete convolution g_j

is defined by

$$g_j = \sum_{k=-\infty}^{\infty} h_{j-k} q_k \qquad \cdot 1$$

·14 Circular convolution

Given two sequences h_j, q_j $j = 0(1)N-1$ their circular or cyclic convolution g'_j is defined by

$$g'_j = \sum_{k=0}^{N-1} h_{(j-k)|N}\, q_k \qquad \cdot 2$$

N.B. If the sequences h_j and q_j of §·13 are such that

$$h_j = 0 \qquad \text{for } j < 0 \text{ or } j > a-1$$

$$q_j = 0 \qquad \text{for } j < 0 \text{ or } j > b-1$$

then $g'_j = g_j$ for $j = 0(1)N-1$ provided that $N \geq a+b-1$.

15 Discrete Fourier transform

Given a set of numbers f_j $j = 0(1)N-1$, the set F_k $k = 0(1)N-1$ defined by

$$F_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{-2\pi ijk/N} f_j$$

is called the discrete Fourier transform (D.F.T.) of f_j. The inverse relationship is

$$f_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi ijk/N} F_k$$

These two mappings are referred to as the forward and inverse or reverse transforms. The $N \times N$ matrix $\underline{\underline{W}}$ is defined by

$$(\underline{\underline{W}})_{jk} = \frac{1}{\sqrt{N}} e^{-2\pi ijk/N}$$

whence $\underline{\underline{W}}^{-1}$ exists and has elements

$$(\underline{\underline{W}}^{-1})_{jk} = \frac{1}{\sqrt{N}} e^{2\pi ijk/N}$$

Defining the vectors $\underline{f}$ and $\underline{F}$ by $(\underline{f})_j = f_j$ and $(\underline{F})_j = F_j$ then the above relations will be written

$$\underline{F} = \underline{\underline{W}}\,\underline{f} \quad \text{and} \quad \underline{f} = \underline{\underline{W}}^{-1}\underline{F}$$

·16 Miscellaneous

An $N \times N$ matrix $\underline{\underline{h}}$ whose j,k th element is $(\underline{\underline{h}})_{jk} = h_{(j-k)|N}$ ($j,k = 0(1)N-1$ where $\{h_0,\ldots,h_{N-1}\}$ is a set of real or complex numbers) is called a circulant matrix. In such a matrix each column (row) contains the same elements as the one on its left (above) but shifted one place down (to the right).

Given a sequence of N real or complex numbers $x_0,\ldots,x_{N-1}$ the following are defined

a vector $\underline{x} = (x_0,\ldots,x_{N-1})'$

a matrix $\underline{\underline{x}}$ such that $(\underline{\underline{x}})_{jk} = x_{(j-k)|N}$ so that, by definition, $\underline{x}$ is the first column of the circulant matrix $\underline{\underline{x}}$

a vector $\underline{X} = \underline{\underline{W}}\,\underline{x}$ i.e. the D.F.T. of $\underline{x}$

a matrix $\underline{\underline{X}} = \text{diag}(\underline{X}) = \text{diag}(X_0,\ldots,X_{N-1})$

N.B. Where there is little risk of confusion this notation will occasionally be abused e.g. in §2·1 $\underline{m}$ is a sample from an $N(\underline{\mu}, \text{diag}(\underline{\mu}))$ distribution and it is convenient to use $\underline{\underline{M}}$ to denote $\text{diag}(\underline{\mu})$, but there is no suggestion that $\underline{\underline{M}}$ and $\underline{m}$ are related by $\underline{\underline{M}} = \text{diag}(\underline{\underline{W}}\,\underline{m})$.

·2 Theorems

·21 Theorem

If $\underline{\underline{h}}$ is a circulant matrix then

$$\underline{\underline{W}}\ \underline{\underline{h}}\ \underline{\underline{W}}^{-1} = \sqrt{N}\ \underline{\underline{H}}$$

i.e. since the diagonal elements of $\underline{\underline{H}}$ are the D.F.T. of $\underline{h}$ the product $\underline{\underline{W}}\ \underline{\underline{h}}\ \underline{\underline{W}}^{-1}$ may be calculated by a single D.F.T.

Proof

$$(\underline{\underline{W}}\ \underline{\underline{h}}\ \underline{\underline{W}}^{-1})_{jm} = \sum_{k,\ell=0}^{N-1} (\underline{\underline{W}})_{jk}\ (\underline{\underline{h}})_{k\ell} (\underline{\underline{W}}^{-1})_{\ell m}$$

$$= \sum_{k,\ell} h_{(k-\ell)|N} \frac{1}{\sqrt{N}} e^{-2\pi ijk/N}\ \frac{1}{\sqrt{N}} e^{2\pi i\ell m/N}$$

$$= \frac{1}{\sqrt{N}} \sum_{\ell} \sum_{k} h_{(k-\ell)|N}\ \frac{1}{\sqrt{N}} e^{-2\pi ij(k-\ell)/N} \cdot e^{2\pi i\ell(m-j)/N}$$

$$= \frac{1}{\sqrt{N}} \sum_{\ell} \sum_{n} h_n \frac{1}{\sqrt{N}}\ e^{-2\pi ijn/N} \cdot e^{2\pi i\ell(m-j)/N}$$

$$= \frac{1}{\sqrt{N}} \sum_{n} h_n \frac{1}{\sqrt{N}}\ e^{-2\pi ijn/N} \cdot N\delta_{mj}$$

$$= \sqrt{N}\ (\underline{\underline{W}}\ \underline{h})_j\ \delta_{mj}$$

$$= \sqrt{N}\ (\underline{\underline{H}})_{jm}$$

·22 Corollary

$$\underline{\underline{h}} = \sqrt{N}\ \underline{\underline{W}}^{-1} \underline{\underline{H}}\ \underline{\underline{W}}$$

·23 Theorem

$$\underline{h} \text{ is real } \Leftrightarrow (\underline{H})_j = \overline{(\underline{H})_{N-j}} \qquad j = 1(1)N-1$$

Proof

$$\Rightarrow \qquad (\underline{H})_j = (\underline{\underline{W}}\ \underline{h})_j = \sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} e^{-2\pi ikj/N}\ h_k$$

$$= \sum_{k} \frac{1}{\sqrt{N}}\ e^{2\pi ik(N-j)/N}\ h_k$$

$$= \overline{\sum_k \frac{1}{\sqrt{N}} e^{-2\pi ik(N-j)/N} h_k} \qquad \text{since } h_k = \overline{h_k}$$

$$= \overline{(\underline{\underline{W}}\ \underline{h})_{N-j}}$$

$$= \overline{(\underline{H})_{N-j}} \ .$$

$$\Leftarrow \qquad (\overline{\underline{h}})_j = \overline{(\underline{\underline{W}}^{-1}\underline{H})_j} = \sum_{k=0}^{N-1} \frac{1}{\sqrt{N}} e^{2\pi ijk/N}\, H_k$$

$$= \sum_k \frac{1}{\sqrt{N}} e^{-2\pi ijk/N}\, \overline{H}_k$$

$$= \sum_k \frac{1}{\sqrt{N}} e^{2\pi ij(N-k)/N}\, H_{n-k}$$

$$= \sum_{\ell=0}^{N-1} \frac{1}{\sqrt{N}} e^{2\pi ij\ell/N}\, H_\ell \qquad \text{putting } \ell = N-k.$$

$$= (\underline{\underline{W}}^{-1}\underline{H})_j$$

$$= (\underline{h})_j$$

·24 Corollary

From an almost identical proof $(\underline{h})_j = \overline{(\underline{h})_{N-j}} \Leftrightarrow \underline{H}$ is real

·25 Definition

An N vector $\underline{h}$ is <u>symmetric</u> if $(\underline{h})_j = (\underline{h})_{N-j}$ for $j = 1(1)N-1$

26 Corollary

$\underline{h}$ is real and symmetric $\Leftrightarrow$ $\underline{H}$ is real and symmetric

Proof

follows from Th·23 , Cor·24 , and Defn·25 .

With the definitions given in §·15 the circular convolution of E·2 can be written

$\underline{g} = \underline{\underline{h}}\ \underline{q}$ provided $N \geq a + b - 1$ with a,b as in §·14

·27 Theorem (Convolution Theorem)

$$\underline{g} = \underline{\underline{h}}\ \underline{g} \iff \underline{G} = \sqrt{N}\ \underline{\underline{H}}\ \underline{Q}$$

Proof

$$\Rightarrow \quad \underline{G} = \underline{\underline{W}}\ \underline{g} = \underline{\underline{W}}\ \underline{\underline{h}}\ \underline{g} = \underline{\underline{W}}\ \underline{\underline{h}}\ \underline{\underline{W}}^{-1}.\ \underline{\underline{W}}\ \underline{g} = \sqrt{N}\ \underline{\underline{H}}\ \underline{Q} \quad \text{from Th·21}$$

$$\Leftarrow \quad \underline{g} = \underline{\underline{W}}^{-1}\underline{G} = \underline{\underline{W}}^{-1}\sqrt{N}\ \underline{\underline{H}}\ \underline{Q} = \sqrt{N}\ \underline{\underline{W}}^{-1}\underline{\underline{H}}\ \underline{\underline{W}}\ .\ \underline{\underline{W}}^{-1}\underline{Q} = \underline{\underline{h}}\ \underline{g} \quad \text{from Cor·22}$$

·28 Theorem (Parseval's Theorem

Given a sequence of N numbers $\underline{x}$ and its D.F.T. $\underline{X}$ then

$$\underline{x}^T\underline{x} = \underline{X}^T\underline{X} \quad .$$

Proof

$$\begin{aligned} \underline{X}^T\underline{X} &= (\underline{\underline{W}}\ \underline{x})^T(\underline{\underline{W}}\ \underline{x}) \\ &= \underline{x}^T\ \underline{\underline{W}}^T\ \underline{\underline{W}}\ \underline{x} \\ &= \underline{x}^T\underline{x} \end{aligned}$$

since $\underline{\underline{W}}^T = \underline{\underline{W}}^{-1}$ from the definition of $\underline{\underline{W}}$.

Note that if $\underline{x}$ is real this becomes $\underline{x}'\underline{x} = \underline{X}^T\underline{X}$.

·29 Theorem (Auto-correlation theorem)

$$\underline{\underline{W}}\ \underline{\underline{x}}'\underline{x} = \sqrt{N}\ \ \underline{\underline{X}}^T\underline{X}$$

Proof

$$\begin{aligned} \underline{\underline{W}}\ \underline{\underline{x}}'\underline{x} &= \underline{\underline{W}}\ \underline{\underline{x}}'\underline{\underline{W}}^{-1}\ \underline{\underline{W}}\ \underline{x} \\ &= (\underline{\underline{W}}^{-1T}\underline{\underline{x}}'^T\ \underline{\underline{W}}^T)^T\underline{X} \\ &= (\underline{\underline{W}}\ \underline{\underline{x}}\ \underline{\underline{W}}^{-1})^T\ \underline{X} \\ &= \sqrt{N}\ \underline{\underline{X}}^T\underline{X} \end{aligned}$$

·3 Relationship between D.F.T. and C.F.T.

·31 Function with support on $[0,(N - 1)\Delta r]$

Let $f(x)$ and $F(X)$ be a C.F.T. pair and let f_j and F_k be a D.F.T. pair and put $f_j = f(j\Delta x)$. A relationship is to be indicated between F_k and $F(X)$.

A relationship is first developed between the C.F.T. of $f(x)$ and the C.F.T. of the sampled function $\sum_{j=-\infty}^{\infty} f(x)\delta(x - j\Delta x)$. It is supposed that $f(x) \to 0$ as $|x| \to \infty$ and that $F(x) \to 0$ as $|X| \to \infty$.

From (Bracewell, 1965 : ch. 10), the C.F.T. of

$$\sum_{j=-\infty}^{\infty} \delta(x - j\Delta x) \quad \text{is} \quad \Delta x^{-1} \sum_{k=-\infty}^{\infty} \delta\left(X - \frac{k}{\Delta x}\right) .$$

The integral $I = \int_{-\infty}^{\infty} \left\{ \sum_{j=-\infty}^{\infty} f(x)\delta(x - j\Delta x) \right\} e^{-2\pi i x X} dx$

can be regarded in two different ways:- first as the F.T. of the product of two functions i.e.

$$I = \int_{-\infty}^{\infty} \left\{ f(x) \; . \; \sum_j \delta(x - j\Delta x) \right\} e^{-2\pi i x X} dx$$

$$= F(X) * \Delta x^{-1} \sum_{\ell = -\infty}^{\infty} \delta(X - \ell/\Delta x)$$

$$= \Delta x^{-1} \sum_{\ell = -\infty}^{\infty} F(X - \ell/\Delta x) \qquad \cdot 1$$

and second as the sum of a series of numbers i.e.

$$I = \int_{-\infty}^{\infty} \{f(x) e^{-2\pi i x X}\} \sum_{j=-\infty}^{\infty} \delta(x - j\Delta x) dx$$

$$= \sum_j \int_{-\infty}^{\infty} \{f(x) e^{-2\pi i x X}\} \delta(x - j\Delta x) dx$$

$$= \sum_{j=-\infty}^{\infty} f(j\Delta x) e^{-2\pi i j \Delta x X} \qquad \cdot 2$$

Equating E·1 and E·2

$$\Delta x^{-1} \sum_{\ell = -\infty}^{\infty} F(X - \ell \Delta x^{-1}) = \sum_{j=-\infty}^{\infty} f_j \; e^{-2\pi i j \Delta x X}$$

and, putting $X = k\,N^{-1}\Delta x^{-1}$ for $0 \le k \le N - 1$

$$\Delta x^{-1} \sum_{\ell=-\infty}^{\infty} F((k - N\ell)N^{-1}\Delta x^{-1}) = \sum_{j=-\infty}^{\infty} f_j\, e^{-2\pi ijk/N} \qquad \cdot 3$$

$$= \sum_{j=0}^{N-1} f_j\, e^{-2\pi ijk/N} \qquad \cdot 4$$

provided $f(x) = 0$ for $x < 0$ and $x > (N - 1)\Delta x$

$$= \sqrt{N}\, F_k \;.$$

Hence $$F_k = \frac{1}{\Delta x\sqrt{N}} \sum_{\ell} F((k - \ell N)N^{-1}\Delta x^{-1}) \qquad \cdot 5$$

If $F(X) = 0\ \forall |X| \ge (2\Delta x)^{-1}$, then manipulation of E·5 shows that

$$F_k = \begin{cases} \dfrac{1}{\Delta x\sqrt{N}} F(kN^{-1}\Delta x^{-1}) & 0 \le k \le N/2 \\[2ex] \dfrac{1}{\Delta x\sqrt{N}} F((k - N)N^{-1}\Delta x^{-1}) & N/2 \le k \le N - 1 \end{cases} \qquad \cdot 6$$

Note that defining ΔX to be the increment in X, then $X = k\,N^{-1}\Delta x^{-1}$ leads to $\Delta X = N^{-1}\Delta x^{-1}$ or

$$\Delta X \Delta x\, N = 1 \qquad \cdot 7$$

Precisely the same situation holds in reverse if $F(X)$ is the starting point and one defines $F_k = F(k\Delta X)$ and then compares $f(x)$ with f_j. In this case

$$f_j = \frac{1}{\Delta X\sqrt{N}} \sum_{\ell} f((j - \ell N)N^{-1}\Delta X^{-1}) \qquad \cdot 8$$

$$= \Delta x\sqrt{N} \sum_{\ell} f((j - \ell N)\Delta x) \qquad \cdot 9$$

·32 Function with support in $\left[-\frac{N-1}{2}\Delta x,\ \frac{N-1}{2}\Delta x\right]$

Following precisely the same outline as §·31 but with the transform pair

$$\sum_{j=-\infty}^{\infty} \delta\left(x - (j - \frac{N-1}{2})\Delta x\right) \quad \text{and} \quad \Delta x^{-1} \sum_{k=-\infty}^{\infty} e^{\pi ik(N-1)} \delta(X - k\Delta x^{-1})$$

and defining I by

$$I = \int_{-\infty}^{\infty} \left\{ \sum_{j=-\infty}^{\infty} f(x)\ \delta\left(x - (j - \frac{N-1}{2})\Delta x\right) \right\} e^{-2\pi ixX} dx$$

then

$$I = F(X) * \Delta x^{-1} \sum_{\ell=-\infty}^{\infty} e^{\pi i\ell(N-1)} \delta(X - k\Delta x^{-1})$$

$$= \Delta x^{-1} \sum_{\ell} e^{\pi i\ell(N-1)} F(x - \ell\Delta x^{-1}) \qquad \cdot 1$$

$$\text{and} \quad I = \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ f(x)\ e^{-2\pi ixX} \right\} \delta\left(x - (j - \frac{N-1}{2})\Delta x\right) dx$$

$$= \sum_{j} f\left((j - \frac{N-1}{2})\Delta x\right) e^{-2\pi iX\Delta x(j - \frac{N-1}{2})} \quad . \qquad \cdot 2$$

$$\text{Hence} \quad \Delta x^{-1} \sum_{\ell} e^{\pi i\ell(N-1)} F(X - \ell\Delta x^{-1}) = \sum_{j} f((j - \frac{N-1}{2})\Delta x) e^{-2\pi iX\Delta x(j - \frac{N-1}{2})}$$

and putting $X = k\,N^{-1}\Delta x^{-1}$ for $0 \le k \le N-1$

$$\Delta x^{-1} \sum_{\ell} e^{\pi i\ell(N-1)} F((k - \ell N)N^{-1}\Delta x^{-1}) = \sum_{j=-\infty}^{\infty} f_j\ e^{-2\pi ijk/N}\ e^{\pi ik(1-N^{-1})}$$

$$= e^{\pi ik(1-N^{-1})} \sum_{j=0}^{N-1} f_j e^{-2\pi ijk/N}$$

provided $f(x) = 0\ \forall |x| > \frac{N-1}{2}\Delta x$

$$= e^{\pi ik(1-N^{-1})} \sqrt{N}\, F_k \quad .$$

Hence

$$F_k = \frac{1}{\Delta x\sqrt{N}} \sum_{\ell} e^{-\pi i(1-N^{-1})(k-\ell N)} F((k-\ell N)N^{-1}\Delta x^{-1}) \qquad \cdot 5$$

and if $F(X) = 0\ \forall |X| \ge (2\Delta x)^{-1}$, then

$$F_k = \begin{cases} \dfrac{1}{\Delta x\,\sqrt{N}}\; e^{-\pi i(1-N^{-1})k}\; F(kN^{-1}\Delta x^{-1}) & 0 \le k \le N/2 \\ \dfrac{1}{\Delta x\,\sqrt{N}}\; e^{-\pi i(1-N^{-1})(k-N)}\; F((k-N)N^{-1}\Delta x^{-1}) & \frac{N}{2} \le 1 \le N-1 \end{cases} \qquad \cdot 6$$

·4 The Radon transform and its properties

·41 Introduction

In this section the basic properties of the Radon transform are given. There are, apart from various definitions and minor theorems required along the way, three principal results:-

a) The Projection theorem

b) The Back projection theorem

c) Proof of the discontinuity of $\mathcal{R}^{-1}$

The Projection theorem is implicit in all Fourier based techniques for evaluating inverse Radon transforms, in practice however it is more common to implement the inversion by using an algorithm based on the Back projection theorem. The final theorem given is of fundamental significance to this thesis and is the root cause of the presence of optimization techniques found in Chapter 2.

·42 Notation in §·4

In this section the notation used is not always in line with that used elsewhere, e.g. vectors are not underlined. Below is given a list of the principal symbols used.

Point sets

$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$

$B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ N.B. $B^1 = [-1:1]$

$A\backslash B$: the complement of B in A

$\mathbb{C}$: the set of all complex numbers

Points

r, R in $\mathbb{R}^1$

ϕ in S^{n-1}

x, X, ξ in $\mathbb{R}^n$

Function sets

$C(B^n)$ all functions defined on $\mathbb{R}^n$, continuous on B^n (and from the interior for $|x| = 1$) and zero on $\mathbb{R}^n \setminus B^n$

$B(B^n)$ all functions defined on $\mathbb{R}^n$, bounded on B^n and zero on $\mathbb{R}^n \setminus B^n$.

$\mathcal{L}^2(B^n)$ all functions defined on $\mathbb{R}^n$, zero on $\mathbb{R}^n \setminus B^n$ and whose squared modulus is Lebesgue integrable.

Measures

$d\phi^\perp$ Lebesgue measure on $\{x \in \mathbb{R}^n : \langle x,\phi\rangle = r\}$

dr Lebesgue measure on $\mathbb{R}$

$d\phi$ Lebesgue measure on S^{n-1}

dx Lebesgue measure on $\mathbb{R}^n$

$m(\Omega)$ the measure of a set Ω

S^{n-1} the measure of the set S^{n-1}

B^n the measure of the set B^n

Miscellaneous

$\langle x',x''\rangle$ inner product of x' and x'' on $\mathbb{R}^n$

$\|f\|_{X,\omega}$ $\int_X |f|^2\, \omega dm$ where m is the appropriate measure on X..

$d(x',x'')$ metric on $\mathbb{R}^n$

See also §·47

·43 Definition of Radon transform

Given a function f its Radon transform is denoted by $\mathcal{R}f$. There is a noticable divergence of opinion in the literature on Radon transforms concerning the domain of definition for f. Broadly speaking the mathematically orientated literature tends to define f on the set $\mathbb{R} \times S^{n-1}$ (i.e. on a surface which is topologically equivalent to a cylinder in $\mathbb{R}^{n+1}$) by

$$(\mathcal{R}f)(r,\phi) = \int_{<x,\phi>=r} f(x)\,d\phi^{\perp} \qquad \forall\ (r,\phi) \in \mathbb{R} \times S^{n-1}$$

while the more applications orientated literature tends to define $\mathcal{R}f$ by

$$(\mathcal{R}f)(r\phi) = \int_{<x,\phi>=r} f(x)\,d\phi^{\perp} \qquad \forall\ \ r\phi\ \ \mathbb{R}^n,\ \ \phi \in S^{n-1},\ r \in \mathbb{R}$$

It is largely a matter of preference though if the second is used there is the question of what value to assign $(\mathcal{R}f)(0)$ as in general the value of the integral will depend on ϕ even if $r = 0$.

Indiscussing the Radon transform in this section, two theorems are given which shape the whole discussion in the remainder of the thesis, they are that $\mathcal{R}$ is continuous and that $\mathcal{R}^{-1}$ exists and is discontinuous. Discussion of the continuity of $\mathcal{R}$ and $\mathcal{R}^{-1}$ presupposes a definition of $\|\mathcal{R}f\|$ and this in turn requires a definition for the domain of $\mathcal{R}f$. In order to show that the future discussion would not be radically changed by a minor change in the definition of $\mathcal{R}$, both possibilities will be borne in mind throughout this section. Thereafter the first definition will be used at all times.

In addition to the two possible domains for $\mathcal{R}f$, we shall also consider the possibility of introducing a weight function $(1 - r^2)^{-(n-1)/2}$ which allows one to make use of certain orthogonality relations when using the polynomials suggested by Marr and others (Marr, 1974). The possible domains of definition, weight functions and Lebesgue measures are summarised in the following table.

domain of $\mathcal{R}f$	weight function	Lebesgue measure
$\mathbb{R}^n$	unit	$m_1 = r^{n-1}dr\ d\phi$
$\mathbb{R} \times S^{n-1}$	unit	$m_2 = dr\ d\phi$
$\mathbb{R}^n$	$\omega_1 = (\|r\| \sqrt{(1 - r^2)})^{-(n-1)}$	$m_3 = (1 - r^2)^{-(n-1)/2}dr\ d\phi$
$\mathbb{R} \times S^{n-1}$	$\omega_2 = (1 - r^2)^{-(n-1)/2}$	$m_4 = (1 - r^2)^{-(n-1)/2}dr\ d\phi$

· 4311 Definition

If $f \in C(B^n)$, the Radon transform $f \in C(B^1 \times S^{n-1})$ is given by

$$(\mathcal{R} f)(r,\phi) = \int_{\langle x,\phi\rangle = r} f(x)\, d\phi^{\perp}$$

Since f is continuous (except possibly on S^{n-1}) this integral exists for all $(r,\phi) \in \mathbb{R} \times S^{n-1}$ and it is easy to see that $\mathcal{R} f$ has support in $B^1 \times S^{n-1}$ and is continuous.

·4312 Definition

If $f \in C(B^n)$, the Radon transform f defined in $\mathbb{R}^n$ is given by

$$(\mathcal{R} f)(r\phi) = \int_{\langle x,\phi\rangle = r} f(x)\, d\phi^{\perp}$$

As in §·4311 this integral exists for all $r\phi \in \mathbb{R}^n$ and has support in B^n, however as pointed out above, it is not (in general) continuous at the origin.

·432 Theorem

If $f \in C(B^n)$ then

$$\sqrt{2}\ \|\mathcal{R}f\|_{\mathbb{R}^n} \le \|\mathcal{R}f\|_{\mathbb{R}\times S^{n-1}} \le \|\mathcal{R}f\|_{\mathbb{R}\times S^{n-1},\omega_2} = \sqrt{2}\,\|\mathcal{R}f\|_{\mathbb{R}^n,\omega_1}$$

Proof

$$2\,\|\mathcal{R}f\|^2_{\mathbb{R}^n} = 2 \int_{S^{n-1}} \int_0^1 |\mathcal{R}f|^2\, r^{n-1} dr\, d\phi$$

$$\le 2 \int_{S^{n-1}} \int_0^1 |\mathcal{R}f|^2\, dr\, d\phi \qquad \text{since } |\mathcal{R}f|^2 \ge 0 \text{ and } 0 \le r^{n-1} \le 1$$

$$= \int_{S^{n-1}} \int_{-1}^1 |\mathcal{R}f|^2\, dr\, d\phi \quad \text{since } (\mathcal{R}f)(r,\phi) = (\mathcal{R}f)(-r,-\phi)$$

$$= \|\mathcal{R}f\|^2_{\mathbb{R}\times S^{n-1}}$$

$$\|\mathcal{R}f\|^2_{\mathbb{R}\times S^{n-1}} = \int_{S^{n-1}} \int_{-1}^{1} |\mathcal{R}f|^2 dr\, d\phi$$

$$= \int_{S^{n-1}} \int_{-1}^{1} |\mathcal{R}f|^2 \frac{(1-r^2)^{\frac{n-1}{2}}}{(1-r^2)^{\frac{n-1}{2}}} dr\, d\phi$$

$$= \int_{S^{n-1}} \int_{-1}^{1} |\mathcal{R}f|^2 \frac{1}{(1-r^2)^{\frac{n-1}{2}}} dr\, d\phi \text{ since } |\mathcal{R}f|^2 \geq 0 \text{ and } 0 \leq (1-r^2)^{\frac{n-1}{2}} \leq 1$$

$$= \|\mathcal{R}f\|^2_{\mathbb{R}\times S^{n-1},\omega_2}$$

$$\|\mathcal{R}f\|^2_{\mathbb{R}\times S^{n-1},\omega_2} = \int_{S^{n-1}} \int_{-1}^{1} |\mathcal{R}f|^2 \frac{1}{(1-r^2)^{\frac{n-1}{2}}} dr\, d\phi$$

$$= 2 \int_{S^{n-1}} \int_{0}^{1} |\mathcal{R}f|^2 \frac{1}{(1-r^2)^{\frac{n-1}{2}}} dr\, d\phi \quad \text{since } (\mathcal{R}f)(r,\phi) = (\mathcal{R}f)(-r,-\phi)$$

$$= 2 \int_{S^{n-1}} \int_{0}^{1} |\mathcal{R}f|^2 \frac{1}{(|r|(1-r^2)^{\frac{1}{2}})^{n-1}} |r|^{n-1} dr\, d\phi$$

$$= 2\,\|\mathcal{R}f\|^2_{\mathbb{R}^n,\omega_1}$$

N.B. When $f \in C(B^1 \times S^{n-1})$ then the integrals defining $\|\mathcal{R}f\|_{\mathbb{R}\times S^{n-1}}$ and $\|\mathcal{R}f\|_{\mathbb{R}\times S^{n-1},\omega_2}$ are both well defined since $\mathcal{R}f$ is a continuous function with compact support. The same cannot be said when $\mathcal{R}f$ is considered as defined on R^n - see §·41.

·433 Theorem

The operator $\mathcal{R} : C(B^n) \to C(B^1 \times S^{n-1},\omega_2)$ is a bounded and linear.

Proof

a) $\mathcal{R}$ is linear - obvious.

b) $$\|\mathcal{R}f\|^2_{\mathbb{R}\times S^{n-1},\omega_2} = \int_{S^{n-1}} d\phi \int_{-1}^{1} \frac{dr}{(1-r^2)^{\frac{n-1}{2}}} \,|f|^2$$

$$= \int_{S^{n-1}} d\phi \int_{-1}^{1} \frac{dr}{(1-r^2)^{\frac{n-1}{2}}} \left| \int_{\langle x,\phi\rangle = r} f(x)\, d\phi^{\perp} \right|^2$$

$$\leq \int_{S^{n-1}} d\phi \int_{-1}^{1} \frac{dr}{(1-r^2)^{\frac{n-1}{2}}} \left\{ \int_{\langle x,\phi\rangle = r} |f| \quad d\phi^{\perp} \right\}^2$$

$$\leq \int_{S^{n-1}} d\phi \int_{-1}^{1} \frac{dr}{(1-r^2)^{\frac{n-1}{2}}} \left\{ \int_{\substack{\langle x,\phi\rangle = r \\ \cap B^n}} d\phi^{\perp} \int_{\langle x,\phi\rangle = r} d\phi^{\perp} |f|^2 \right\}$$

by Hölder's inequality

$$= \int_{S^{n-1}} d\phi \int_{-1}^{1} \frac{dr}{(1-r^2)^{\frac{n-1}{2}}} \; B^{n-1}(1-r^2)^{\frac{n-1}{2}} \int_{\langle x,\phi\rangle = r} d\phi^{\perp} \,|f|^2$$

$$= B^{n-1} \int_{S^{n-1}} d\phi \int_{-1}^{1} dr \int_{\langle x,\phi\rangle = r} d\phi^{\perp} |f|^2$$

$$= B^{n-1} \int_{S^{n-1}} d\phi \, \|f\|^2_{\mathbb{R}^n}$$

$$= B^{n-1} S^{n-1} \, \|f\|^2_{\mathbb{R}^n}$$

hence $\|\mathcal{R}f\|_{\mathbb{R}\times S^{n-1},\omega_2} \leq \sqrt{B^{n-1} S^{n-1}} \; \|f\|_{\mathbb{R}^n}$

·434 Corollary

The operators
$$\mathcal{R}: C(B^n) \to B(B^n)$$
$$\mathcal{R}: C(B^n) \to C(B^1 \times S^{n-1})$$
$$\mathcal{R}: C(B^n) \to B(B^n, \omega_1)$$

are linear and bounded.

Proof

This follows immediately from Theorems ·432 and ·433.

·435 Theorem: Extension by continuity

If X_0 is a dense linear subspace of the normal space X and T_0 is a bounded operater from X_0 into the Banach space Y, then T_0 has a unique extension to a bounded operator $T : X \to Y$ such that $\|T\| = \|T_0\|$.

Proof

A proof in different notation is given in Bochner (Bochner et al, 1949: p 92, Th. 47).

·436 In §·431 $\mathcal{R}$ was defined on the set $C(B^n)$, in §·432 - §·434 it was shown that for several different definitions the operator $\mathcal{R}$ was continuous. These results are now used with the theorem of §·435 to provide a definition of $\mathcal{R}$ on the set $\mathcal{L}^2(B^n)$.

Note first that $C(B^n)$ is a dense linear subspace of $\mathcal{L}^2(B^n)$ (Radon, 1974: Theorem 3·14) and that $\mathcal{L}^2(B^1 \times S^{n-1})$ and $\mathcal{L}^2(B^n)$ are Banach spaces which ever of the measures given in §·43 is used (Rudin, 1974: Theorem 3·11). Since it was shown in Theorems ·433 and ·434 that $\mathcal{R}$ is continuous regardless of which of the four view points is taken, it follows that all the conditions of Thoerem ·435 are satisfied.

Definition

The mappings
$$\mathcal{R}: \mathcal{L}^2(B^n) \to \mathcal{L}^2(B^n)$$
$$\mathcal{R}: \mathcal{L}^2(B^n) \to \mathcal{L}^2(B^1 \times S^{n-1})$$
$$\mathcal{R}: \mathcal{L}^2(B^n) \to \mathcal{L}^2(B^n, \omega_1)$$
$$\mathcal{R}: \mathcal{L}^2(B^n) \to \mathcal{L}^2(B^1 \times S^{n-1}, \omega_2)$$
are defined to be the "extension by continuity" of the mappings $\mathcal{R}$ on $C(B^n)$ discussed in §·43 to ·434.

·44 The Projection Theorem

In this section the well known result relating the Fourier transform of a function to its Radon transform is given. To obtain this result $\mathcal{R}$ will be considered as a mapping from $\mathcal{L}^2(B^n)$ to $\mathcal{L}^2(\mathbb{R} \times S^{n-1})$, however the result remains true for any of the four definitions of .

·441 Definition and Properties of $\mathcal{F}_n$

·4411 Definition of $\mathcal{F}_n$

Given $f \in \mathcal{L}^2(\mathbb{R}^n)$ the operator $\mathcal{F}_n : \mathcal{L}^2(\mathbb{R}^n) \to \mathcal{L}^2(\mathbb{R}^n)$ is defined by

$$(\mathcal{F}_n f)(X) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i < x, X>} dx$$

·4412 Definition of $\mathcal{F}_n^{-1}$

Given $f \in \mathcal{L}^2(\mathbb{R}^n)$ the operator $\mathcal{F}_n^{-1}$ is defined by

$$f(x) = \mathcal{F}_n^{-1}(\mathcal{F}_n f)(x) = \int_{\mathbb{R}^n} (\mathcal{F}_n f)(X) e^{2\pi i < X, x>} dX$$

·4413 Parseval's Theorem

If $f \in \mathcal{L}^2(\mathbb{R}^n)$ then $\mathcal{F}_n f \in \mathcal{L}^2(\mathbb{R}^n)$ and $||\mathcal{F}_n f||_{\mathbb{R}^n} = ||f||_{\mathbb{R}^n}$

·4414 Discussion

In fact the definitions given in §§·4411 - ·4412 are over simplified. If $f \in \mathcal{L}^2$ then in general the integral will not exist. More correctly $\mathcal{F}_n$ is defined by the above formulae for $f \in \mathcal{L}^2 \cap \mathcal{L}^1$, shown to be continuous, and then abstractly extended to all of $\mathcal{L}^2$. However, the definitions given above are sufficient for the present work. Note that $\mathcal{F}_n$ is unitary so that $\mathcal{F}_n$ and $\mathcal{F}_n^{-1}$ are both continuous. For further details see Bochner (Bochner et al, 1949: Ch 4).

·4415 In this section a theorem is proved which is not of immediate interest but is required later in the proof that $\mathcal{P}$ (yet to be defined) is continuous.

Theorem

The set $\mathcal{F}_n(\mathcal{L}^2(B^n)) \cap \{f: ||f|| \le 1\}$ is equicontinuous.

Proof

If $f \in \mathcal{L}^2(B^n)$ then

$$|(\mathcal{F}_n f)(X + \Delta X) - (\mathcal{F}_n f)(X)| = \left| \int_{B^n} f(x) e^{-2\pi i < x,X>} (e^{-2\pi i < x,\Delta X>} - 1)\, dx \right|$$

$$\le \int_{B^n} |f(x)| \left| e^{-2\pi i < x,\Delta X>} - 1 \right| dx$$

$$= \int_{B^n} |f(x)| \; |2 \sin(\pi < x,\Delta X>)| \; dx$$

$$\le \left\{ \int_{B^n} |f(x)|^2 dx \int_{B^n} |2 \sin(\pi <x,\Delta X>)|^2 \, dx \right\}^{\frac{1}{2}}$$ by using Holder's inequality

$$\le ||f|| \left\{ \int_{B^n} |2\pi \Delta X|^2 dx \right\}^{\frac{1}{2}}$$

$$= 2\pi \; |\Delta X| \sqrt{B^n} \; ||f||$$

and if $||f|| \le 1$ then given any $\varepsilon > 0$ by putting $\delta = \varepsilon/(2\pi\sqrt{B^n})$ it follows that

$$|X| < \delta \Rightarrow |(\mathcal{F}_n f)(X + \Delta X) - (\mathcal{F}_n f)(X)| < \varepsilon$$

·442 Definition and properties of $\mathcal{F}_1$

·4421 Definition of $\mathcal{F}_1$ on $\mathcal{L}^2(\mathbb{R})$

From §·4411 $\mathcal{F}_1$ is defined on $\mathcal{L}^2(\mathbb{R})$

·4422 Definition of $\mathcal{F}_1$ on $\mathcal{L}^2(\mathbb{R} \times S^{n-1})$

Given $f(r,\phi) \in \mathcal{L}^2(\mathbb{R} \times S^{n-1})$, the function $(\mathcal{F}_1 f)$ is defined by regarding ϕ as a constant parameter i.e. by Fourier transformation with respect to the first variable only. Thus $\mathcal{F}_1 : \mathcal{L}^2(\mathbb{R} \times S^{n-1}) \to \mathcal{L}^2(\mathbb{R} \times S^{n-1})$ is given by

$$(\mathcal{F}_1 f)(R,\phi) = \int_{\mathbb{R}} f(r,\phi)e^{-2\pi i r R}dr$$

·4423 Parseval's Theorem

From §·4413, the relationship $||\mathcal{F}_1 f||_{\mathbb{R}} = ||f||_{\mathbb{R}}$ immediately follows but now both sides of this equation are functions of the parameter ϕ. Integrating the square of both sides with respect to ϕ gives the modified relation

$$||\mathcal{F} f||_{\mathbb{R} \times S^{n-1}} = ||f||_{\mathbb{R} \times S^{n-1}} \quad \text{for} \quad f \in \mathcal{L}^2(\mathbb{R} \times S^{n-1})$$

·4424 Discussion

Note that $\mathcal{F}_1$, whether defined on $\mathcal{L}^2(\mathbb{R})$ or $\mathcal{L}^2(\mathbb{R} \times S^{n-1})$, has the obvious inverse indicated by §·4412 and is again unitary so that $\mathcal{F}_1$ and $\mathcal{F}_1^{-1}$ are both continuous. The qualifications of §·4414 are also applicable to this definition.

·443 Definition and properties of $\mathcal{P}$

·4431 Definition of $\mathcal{P}$

Given $F(R\phi)$ defined on $\mathbb{R}^n$ then $\mathcal{P}F$ defined on $\mathbb{R} \times S^{n-1}$ is given by

$$(\mathcal{P}F)(R,\phi) = F(R\phi).$$

·4432 Discussion

In §·444 the result that $\mathcal{P}$ is continuous will be required. In view of the simplicity of $\mathcal{P}$ one might think that this would be a fairly straightforward result to obtain. This does not appear to be the case, even the assumption of continuity for F is not sufficient. As will be seen it is necessary to restrict F to being a member of a set whose elements are uniformly continuous and in which a bounded subset is equicontinuous and even then the result is rather tedious to obtain.

In the following sections $\mathcal{P}$ will be shown to be continuous when its domain is taken as $\mathcal{F}_n(\mathcal{L}^2(B^n))$, and continuity will be proved by showing that $\mathcal{P}$ is continuous at 0 i.e. given $\{F_j\}$ such that $\|F_j\|_{\mathbb{R}^n} \to 0$ then $\|\mathcal{P}F_j\|_{\mathbb{R}\times S^{n-1}} \to 0$. Continuity on the whole domain then follows from standard theorems (Radon, 1974: Th 5 4 (c)). It will be assumed that $\|F_j\| \le 1$ for all j, if this is not the case sufficient terms may be discarded at the beginning of the sequence and the convergence properties will remain the same. Throughout the following sections §·4433 - §·4435 it will be assumed that $\{F_j\} \subset {}_n(C(B^n)) \subset \mathcal{L}^2(\mathbb{R}^n)$.

·4433 Example

The following example shows that restricting the domain of $\mathcal{P}$ to $C(B^n)$ is not sufficient to ensure the continuity of $\mathcal{P}$. Define the function $F_j \in C(B^n)$ for $j = 1(1)\infty$ by

$$F_j(R\phi) = \begin{cases} \left[\dfrac{n(n+1)}{S^{n-1}}\, j^n(1 - j|R|)\right]^{\frac{1}{2}} & |R| \le j^{-1} \\ 0 & |R| > j^{-1} \end{cases}$$

Then $\|F_j\|_{\mathbb{R}^n} = \int_0^\infty dR \int_{S^{n-1}} d\phi\ |F_j(R\phi)|^2 R^{n-1} = 1$

and $$\|\mathcal{P}F_j\|_{\mathbb{R}\times S^{n-1}} = \int_{-\infty}^{\infty} dR \int_{S^{n-1}} d\phi\ |F_j(R\phi)|^2$$
$$= n(n+1)j^{n-1}$$

And as $j \to \infty$, $\|\mathcal{P}F_j\|_{\mathbb{R}\times S^{n-1}} \to \infty$ so that $\mathcal{P}$ is discontinuous on $C(B^n)$

·4434 Miscellaneous Lemmas

Lemma 1

If $\{F_j\}$ is equi-continuous and $\|F_j\| \to 0$ then $F_j \to 0$ pointwise a.e..

Proof

Suppose not, then for some ξ, $F_j(\xi)$ is not convergent, hence $\exists\, \varepsilon$ and $\{F_{j_k}(\xi)\} \ni |F_{j_k}(\xi)| > 2\varepsilon$ for all sufficiently large k.

Since $\{F_{j_k}\}$ is equi-continuous $\exists\, \delta > 0 \ni$

$$|\xi - X| < \delta \Rightarrow |F_{j_k}(X) - F_{j_k}(\xi)| < \varepsilon \quad \forall j_k .$$

Thus
$$\begin{aligned} |F_{j_k}(X)| &= |F_{j_k}(\xi) + (F_{j_k}(X) - F_{j_k}(\xi))| \\ &\geq \Big| |F_{j_k}(\xi)| - |F_{j_k}(X) - F_{j_k}(\xi)| \Big| \\ &> 2\varepsilon - \varepsilon = \varepsilon. \end{aligned}$$

Hence
$$\begin{aligned} \|F_{j_k}\|^2 &= \int_{\mathbb{R}^n} |F_{j_k}|^2 \, dX \\ &\geq \int_{|\xi - X| < \delta} |F_{j_k}|^2 \, dX \\ &> \varepsilon \int_{|\xi - X| < \delta} dX \\ &= \varepsilon\, \delta^n B^n \end{aligned}$$

which contradicts the fact that $\|F_j\| \to 0$.

Lemma 2

If $\{F_j\}$ is equi-continuous and $F_j \to 0$ pointwise a.e. then $F_j \to 0$ pointwise.

Proof

Denote by N the set on which F_j is not known to converge pointwise and consider some $X \in N$. Since $m(N) = 0$, every δ neighbourhood of X intersects $\mathbb{R}^n \setminus N$.

Now consider some ε. Since $\{F_j\}$ is equi-continuous $\exists\,\delta > 0 \ni$

$$|\xi - X| < \delta \Rightarrow |F_j(\xi) - F_j(X)| < \varepsilon/2 .$$

Choose $\xi \in \mathbb{R}^n \setminus N$, then $F_j(\xi) \to 0$ pointwise and $\exists j_0 \ni$

$$j > j_0 \Rightarrow |F_j(\xi)| < \varepsilon/2 .$$

Hence for $j > j_0$, $|F_j(X)| = |(F_j(X) - F_j(\xi)) + F_j(\xi)|$

$$\leq |F_j(X) - F_j(\xi)| + |F_j(\xi)|$$

$$< \varepsilon .$$

Hence $F_j(X) \to 0 \quad \forall x \in N$.

Lemma 3

If $F_j \to 0$ pointwise then $\mathcal{P} F_j \to 0$ pointwise

Proof

Trivial.

Lemma 4

If $\{F_j : \mathbb{R}^n \to \mathbb{C}\}$ is equi-continuous then $\{\mathcal{P}F_j : B^1 \times S^{n-1} \to \mathbb{C}\}$ is equi-continuous.

<u>N.B.</u> It is necessary to restrict the domain of $\mathcal{P}F_j$ to $B^1 \times S^{n-1}$. The theorem does not hold for $\mathcal{P}F_j$ defined on $\mathbb{R} \times S^{n-1}$.

Proof

Define metrics on $\mathbb{R} \times S^{n-1}$ and $\mathbb{R}^n$ as follows:-

If (R',ϕ') and (R'',ϕ'') are in $\mathbb{R} \times S^{n-1}$ then

$$d\big((R',\phi'),(R'',\phi'')\big) = \{(R' - R'')^2 + (\cos^{-1} < \phi',\phi'' >)^2\}^{\frac{1}{2}}.$$

If $(R'\phi')$ and $(R''\phi'')$ are in $\mathbb{R}^n$ then

$$d\big((R'\phi'),(R''\phi'')\big) = \{< R'\phi' - R''\phi'', R'\phi' - R''\phi'' >\}^{\frac{1}{2}}.$$

Hence $d_{\mathbb{R} \times S^{n-1}} \geq d_{\mathbb{R}^n} \Leftrightarrow -2R'R'' + (\cos^{-1} < \phi',\phi'' >)^2 \geq -2 R'R'' < \phi',\phi'' >$

$$\Leftrightarrow \quad -2\alpha + \beta^2 \geq -2\alpha \cos \beta$$

where $\alpha = R'R''$ and β is the modulus of the angle between ϕ' and ϕ'', i.e. $0 \leq \beta \leq \pi$. Now

$$\cos \beta = 1 - \frac{\beta^2}{2} + \frac{\beta^3}{6} \sin \theta\beta \qquad 0 < \theta < 1$$

hence for $\beta \geq 0$ $\quad 1 - \frac{\beta^2}{2} + \frac{\beta^3}{6} \geq \cos \beta$ and a sufficient condition for $d_{\mathbb{R} \times S^{n-1}} \geq d_{\mathbb{R}^n}$ is

$$-2\alpha + \beta^2 \geq -2\alpha \{1 - \frac{\beta^2}{2} + \frac{\beta^3}{6}\}$$

or $1 \geq \alpha(1 - \beta/3)$.

If $0 \leq \beta \leq \pi$ then this condition is met by $|\alpha| < 1$, i.e. $|R'|$ and $|R''|$ less than 1.

If $\{F_j\}$ is equi-continuous then this means that

$$\forall e > 0 \quad \exists \delta \ni |X' - X''| < \delta \quad \Rightarrow \quad |F_j(X') - F_j(X'')| < \varepsilon .$$

Let $X' = R'\phi'$ and $X'' = R''\phi''$ then the condition $|X' - X''| < \delta$ may be written

$$d\big((R'\phi'), (R''\phi'')\big) < \delta .$$

Since $d_{\mathbb{R} \times S^{n-1}} \geq d_{\mathbb{R}^n}$ for $|R'| \leq 1$ and $|R''| \leq 1$ the condition

$$d\big((R',\phi'),(R'',\phi'')\big) < \delta$$

ensures that $|X' - X''| < \delta$, so that

$$d((R',\phi')(R'',\phi'')) < \delta \Rightarrow$$

$$|\mathcal{P}F_j(R',\phi') - \mathcal{P}F_j(R'',\phi'')| = |F_j(X') - F_j(X'')| < \varepsilon$$

provided $|R'| \leq 1$ and $|R''| \leq 1$. Hence $\{\mathcal{P}F_j\}$ is equi-continuous on the domain $B^1 \times S^{n-1}$.

Lemma 5

If $P_j \to 0$ pointwise and $\{P_j\}$ is equi-continuous on a set Ω where $m(\Omega) < \infty$ then $P_j \to 0$ uniformly.

(In applying this theorem P_j will be a function $\mathcal{P}F_j$ and $\Omega = B^1 \times S^{n-1}$, but this does not affect the theorem.)

Proof

Define the following sets where $X \in \Omega$:-

$$E_j' = \{X : |P_j(X)| < \varepsilon\} \qquad E_j'' = \{X : |P_j(X)| < \varepsilon/2\}$$

$$D_j' = \bigcap_{k=j}^{\infty} E_k' \qquad D_j'' = \bigcap_{k=j}^{\infty} E_k''$$

then $E_j'' \subset E_j'$, $D_j'' \subset D_j'$. For each $X \in \Omega$

$$\begin{aligned} P_j(X) \to 0 \text{ pointwise} &\Rightarrow |P_j(X)| < \varepsilon/2 \quad \forall \text{ sufficiently large } j \\ &\Rightarrow X \in E_j'' \quad \forall \text{ sufficiently large } j \\ &\Rightarrow X \in D_j'' \quad \forall \text{ sufficiently large } j. \end{aligned}$$

Hence $\{D_j''\}$ and $\{D_j'\}$ are increasing sequences of sets which "expand" to fill all of Ω and

$$m(D_j'') \le m(D_j') \le m(\Omega),$$

$$m(\Omega \setminus D_j'') \ge m(\Omega \setminus D_j')$$

and $m(\Omega \setminus D_j'') \to 0$ provided that $m(\Omega) < \infty$.

If $P_j \to 0$ uniformly then $\forall \varepsilon > 0 \ \exists j_0 \ni$

$$j > j_0 \Rightarrow |P_j(X)| < \varepsilon \qquad \forall X \in \Omega$$

or equivalently

$$j > j_0 \Rightarrow E_j' = \Omega \Leftrightarrow D_j' = \Omega \Leftrightarrow \Omega \setminus D_j' = \emptyset \quad .$$

Suppose this is not the case. Since $\{P_j\}$ is equi-continuous, then given $\varepsilon > 0 \ \exists \delta \ni$

$$|X - \xi| < \delta \Rightarrow |P_j(X) - P_j(\xi)| < \varepsilon/2.$$

Choose j_0 such that $m(\Omega \setminus D''_{j_0}) < m(\{X : |X - \xi| < \delta\})$
and choose ξ such that $\xi \in \Omega \setminus D'_j \subset \Omega \setminus D''_j \quad \forall j > j_0$

Such an ξ exists since it is supposed that $\Omega \setminus D'_j \neq \emptyset$ for any $j > j_0$. Also

$$j > j_0 \Rightarrow m(\Omega \setminus D''_j) < m(\{X : |X - \xi| < \delta\})$$

$$\Rightarrow \{X : |X - \xi| < \delta\} \cap D''_j \neq \emptyset.$$

Hence $\exists X \in \{X : |X - \xi| < \delta\} \cap D''_j \ \forall j > j_0$. Choose X to be such a point. With these choices $\xi \in \Omega \setminus D'_j \subset \Omega \setminus D''_j$, $X \in D''_j$ and $|X - \xi| < \delta$ then

$$\begin{aligned} |P_j(\xi)| &\leq |P_j(\xi) - P_j(X)| + |P_j(X)| \\ &< \epsilon/2 + \epsilon/2 \\ &= \varepsilon \qquad \forall j > j_0 \end{aligned}$$

since $\{P_j\}$ is equi-continuous and $X \in D''_j$. Hence $\xi \in E'_j \ \forall j > j_0$ and so $\xi \in D'_j \ \forall j > j_0$. This contradicts the fact that $\xi \in \Omega \setminus D'_j$ $\forall > j_0$. Hence the supposition that $\Omega \setminus D'_j \neq \emptyset$ for $j > j_0$ is false and the theorem proved.

Lemma 6

If $P_j \to 0$ uniformly on Ω where $m(\Omega) < \infty$ then $\|P_j\| \to 0$.
(In applying this theorem P_j will be a function F_j and $\Omega = B^1 \times S^{n-1}$, but this does not affect the theorem).

Proof

$P_j \to 0$ uniformly means that $\forall (\varepsilon/m(\Omega))^{\frac{1}{2}} \ \exists j_0 \ni$

$$j > j_0 \Rightarrow |P_j(X)| < (\varepsilon/m(\Omega))^{\frac{1}{2}}$$

and $|P_j(X| < (\varepsilon/m(\Omega))^{\frac{1}{2}}$

$$\Rightarrow \int_\Omega |P_j|^2 \leq \left(\frac{\epsilon}{m(\Omega)}\right) m(\Omega) = \varepsilon.$$

·4435 Continuity of $\mathcal{P}$

Theorem

The mapping $\mathcal{P}$ defined on $\mathcal{F}_n(\mathcal{L}^2(B^n))$ is continuous.

Proof

Consider a sequence $\{F_j\} \subset \mathcal{F}_n(\mathcal{L}^2(B^n))$ such that $\|F_j\| \to 0$. Without loss of generality, it will be assumed that $\|F_j\| \leq 1 \quad \forall j$.

$$\begin{aligned}
\|F_j\|^2 &= \int_{\mathbb{R}^n} |F_j|^2 \, dX \\
&= \int_0^\infty R^{n-1} \, dR \int_{S^{n-1}} d\phi \, |F_j|^2 \\
&= \int_0^1 R^{n-1} dR \int_{S^{n-1}} d\phi \, |F_j|^2 + \int_1^\infty R^{n-1} dR \int_{S^{n-1}} d\phi \quad |F_j|^2 \\
&= I_1 + I_2.
\end{aligned}$$

Note: since $m(B^n) < \infty$, $\mathcal{L}^2(B^n) \subset \mathcal{L}^1(B^n)$ hence $\|F_j\|^2$ exists and is finite. (see Bochner et al, 1949: Ch II §6) Also

$$\begin{aligned}
\|\mathcal{P}F_j\|^2 &= \int_{\mathbb{R}\times S^{n-1}} |\mathcal{P}F_j|^2 \quad dR d\phi \\
&= \int_{-\infty}^{\infty} dR \int_{S^{n-1}} d\phi \quad |\mathcal{P}F_j|^2 \\
&= 2 \int_0^\infty dR \int_{S^{n-1}} d\phi \quad |\mathcal{P}F_j|^2 \quad \text{since } (\mathcal{P}F_j)(R,\phi) = F_j(R\phi) = (\mathcal{P}F_j)(-R,-\phi) \\
&= 2 \int_0^1 dR \int_{S^{n-1}} d\phi \quad |\mathcal{P}F_j|^2 + 2 \int_1^\infty dR \int_{S^{n-1}} d\phi \quad |\mathcal{P}F_j|^2 \\
&= 2\, I_3 + 2 I_4.
\end{aligned}$$

Now $\|F_j\| \to 0 \Rightarrow I_1 \to 0$ and $I_2 \to 0$.

(a) Consider I_2 and I_4:

$$
\begin{aligned}
I_2 &= \int_1^\infty R^{n-1} dR \int_{S^{n-1}} d\phi\ |F_j|^2 \\
&\geq \int_1^\infty dR \int_{S^{n-1}} d\phi\ |F_j|^2 \\
&= \int_1^\infty dR \int_{S^{n-1}} d\phi\ |\mathcal{P} F_j|^2 \\
&= I_4
\end{aligned}
$$

Thus $I_2 \to 0 \Rightarrow I_4 \to 0$

(b) Consider I_1 and I_3 : (all Lemmas referred to in this section of the proof are in §·4434)

Since $\{F_j\} \subset \mathcal{F}_n(\mathcal{L}^2(B^n))$ and $\|F_j\| \leq 1$ and since, by Parseval's Theorem $\|F_j\| \leq 1 \Rightarrow \|f_j\| \leq 1$ then §·4415 implies that $\{F_j\}$ is equi-continuous.

$\|F_j\| \to 0$ and $\{F_j\}$ equi-continuous $\Rightarrow$ $F_j \to 0$ pointwise (lemmas 1 and 2)

$F_j \to 0$ pointwise $\Rightarrow \mathcal{P} F_j \to$ pointwise (lemma 3)

$\{F_j\}$ equi-continuous $\Rightarrow$ $\{\mathcal{P}F_j\}$ equi-continuous on $B^1 \times S^{n-1}$ (lemma 4)

$\{\mathcal{P} F_j\}$ equi-continuous on $B^1 \times S^{n-1}$ and $\mathcal{P} F_j \to 0$ pointwise

$\Rightarrow \mathcal{P} F_j \to 0$ uniformly (on $B^1 \times S^{n-1}$) (lemma 5)

on $B^1 \times S^{n-1}$: $\mathcal{P} F_j \to 0$ uniformly $\Rightarrow \|\mathcal{P}F_j\| \to 0$ (lemma 6)

Hence $I_1 \to 0 \Rightarrow I_3 \to 0$.

Finally $\|F_j\| \to 0 \Rightarrow I_1 \to 0$ and $I_2 \to 0$

$\Rightarrow I_3 \to 0$ and $I_4 \to 0$

$\Rightarrow \|\mathcal{P} F_j\| \to 0$.

And $\mathcal{P}$ is continuous (on $\mathcal{F}_n(\mathcal{L}^2(B^n))$) if $\|F_j\| \to 0 \Rightarrow \|\mathcal{P}F_j\| \to 0$.

·4436 Inverse of $\mathcal{P}$

Clearly $\mathcal{P}$ is 1 : 1, hence $\mathcal{P}^{-1}$ exists.

·444 The Projection Theorem

In this section the Projection Theorem is given. This theorem is fundamental to any of the Fourier related reconstruction methods (whether or not they actually use Fourier transforms in their implementation). The theorem is proved first for $f \in C(B^n)$ and continuity is then invoked to extend the result to $\mathcal{L}^2(B^n)$.

Theorem

If $f \in \mathcal{L}^2(B^n)$ then $(\mathcal{P}\mathcal{F}_n f)(R,\phi) = (\mathcal{F}_1\mathcal{R}f)(R,\phi)$ almost everywhere.

Proof

(a) $f \in C(B^n)$.

$$\begin{aligned}(\mathcal{P}\mathcal{F}_n f)(R,\phi) &= (\mathcal{F}_n f)(R\phi) \\ &= \int_{B^n} f(x)e^{-2\pi i\langle x,R\phi\rangle}dx \\ &= \int_{-1}^{1} dr \int_{\langle x,\phi\rangle = r} d\phi^{\perp} \quad f(x)\; e^{-2\pi i r R} \\ &= \int_{-1}^{1} dr\; e^{-2\pi i r R} \int_{\langle x,\phi\rangle = r} f(x)\,d\phi^{\perp} \\ &= (\mathcal{F}_1\mathcal{R}f)(R,\phi)\end{aligned}$$

(b) $f \in \mathcal{L}^2(B^n)$

Note first that $\mathcal{P}, \mathcal{F}_n, \mathcal{F}_1$ and $\mathcal{R}$ are all continuous (§·4435, §·4414, §·4424 and §·433 - §·435) hence the mappings $\mathcal{P}\mathcal{F}_n$ and $\mathcal{F}_1\mathcal{R}$ are both continuous.

Consider any $f \in \mathcal{L}^2(B^n)$. Since $C(B^n)$ is dense in $\mathcal{L}^2(B^n)$ there exists a sequence $\{g_j\} \subset C(B^n)$ such that $g_j \to f$ in the mean. But

$$g_j \to f \text{ in } \mathcal{L}^2 \Rightarrow \mathcal{P}\mathcal{F}_n g_j \to \mathcal{P}\mathcal{F}_n f \quad \text{in } \mathcal{L}^2$$

since $\mathcal{P}\mathcal{F}_n$ is continuous.

And from Rudin (Rudin, 1974 : Th 3·12)

$$\mathcal{P}\mathcal{F}_n g_j \to \mathcal{P}\mathcal{F}_n f \quad \text{in } \mathcal{L}^2$$

$$\Rightarrow \exists \{g_{j_k}\} \subset \{g_j\} \ni \mathcal{P}\mathcal{F}_n g_{j_k} \to \mathcal{P}\mathcal{F}_n f \text{ pointwise a.e..}$$

Clearly $g_{j_k} \to f$ in $\mathcal{L}^2$ hence $\mathcal{F}_1\mathcal{R}g_{j_k} \to \mathcal{F}_1\mathcal{R}f$ in $\mathcal{L}^2$ since $\mathcal{F}_1\mathcal{R}$ is continuous. And from Rudin (Rudin, 1974 : Th3·12)

$$\mathcal{F}_1\mathcal{R}g_{j_k} \to \mathcal{F}_1\mathcal{R}f \text{ in } \mathcal{L}^2$$

$$\Rightarrow \exists \{g_{j_{k_\ell}}\} \subset \{g_{j_k}\} \ni \mathcal{F}_1\mathcal{R}g_{j_{k_\ell}} \to \mathcal{F}_1\mathcal{R}f \text{ pointwise a.e..}$$

Hence $\{g_{j_{k_\ell}}\}$ is such that

$$\mathcal{P}\mathcal{F}_n g_{j_{k_\ell}} \to \mathcal{P}\mathcal{F}_n f \text{ pointwise a.e.}$$

$$\mathcal{F}_1\mathcal{R}g_{j_{k_\ell}} \to \mathcal{F}_1\mathcal{R}f \text{ pointwise a.e.}$$

and $\mathcal{P}\mathcal{F}_n g_{j_{k_\ell}} = \mathcal{F}_1\mathcal{R}g_{j_{k_\ell}}$ since $\{g_{j_{k_\ell}}\} \subset C(B^n)$.

Thus $\mathcal{P}\mathcal{F}_n f = \mathcal{F}_1\mathcal{R}f$ a.e..

·45 Inverse of $\mathcal{R}$

Theorem

$\mathcal{R}^{-1}$ exists and $\mathcal{R}^{-1} = \mathcal{F}_n^{-1}\mathcal{P}^{-1}\mathcal{F}_1$.

Proof

From §·444 $\mathcal{P}\mathcal{F}_n = \mathcal{F}_1\mathcal{R}$ and from §·4424 $\mathcal{F}_1^{-1}$ exists, hence

$$\mathcal{R} = \mathcal{F}_1^{-1}\mathcal{P}\mathcal{F}_n.$$

But from §·4436 and §·4414 both $\mathcal{P}^{-1}$ and $\mathcal{F}_n^{-1}$ exist, hence $\mathcal{R}^{-1}$ exists and is given by

$$\mathcal{R}^{-1} = \mathcal{F}_n^{-1}\mathcal{P}^{-1}\mathcal{F}_1 .$$

·46 The Back Projection Theorem

·461 Definition

Given a function t on $\mathbb{R} \times S^{n-1}$, the back projection of t, denoted by $\mathcal{B}t$ is a function defined on $\mathbb{R}^n$ by

$$(\mathcal{B}t)(x) = \frac{1}{2} \int_{S^{n-1}} t(<x,\phi>,\phi)\,d\phi$$

·462 Theorem

If $f \in \mathcal{L}^2(B^n)$ then $f = \mathcal{B}\mathcal{F}_1^{-1}|R|^{n-1}\mathcal{F}_1\mathcal{R}f$.

Proof

Note first that from the definition of $\mathcal{R}$ $(\mathcal{R}f)(r,\phi) = (\mathcal{R}f)(-r,-\phi)$ and this together with the definition of $\mathcal{F}_1$ gives $(\mathcal{F}_1\mathcal{R}f)(R,\phi) = (\mathcal{F}_1\mathcal{R}f)(-R,-\phi)$.

From §·444 $f = \mathcal{F}_n^{-1}\mathcal{P}^{-1}\mathcal{F}_1\mathcal{R}f$ i.e.

$$f(x) = \int_{\mathbb{R}^n} R^{n-1}dRd\phi \; e^{2\pi i <x,R\phi>}(\mathcal{F}_1\mathcal{R}f)(R,\phi)$$

$$= \int_{S^{n-1}} d\phi \int_0^{\infty} R^{n-1}dR \; e^{2\pi i <x,\phi> R}(\mathcal{F}_1\mathcal{R}f)(R,\phi)$$

$$= \frac{1}{2} \int_{S^{n-1}} d\phi \int_{\infty}^{\infty} dR \; e^{2\pi i<x,\phi>R}|R|^{n-1}(\mathcal{F}_1\mathcal{R}f)(R,\phi)$$

since , f is even

$$= (\mathcal{B}\mathcal{F}_1^{-1}|R|^{n-1}\mathcal{F}_1\mathcal{R}f)(x)$$

where $|R|^{n-1}$ is used to denote both the function " $|R|^{n-1}$ " and the operator " $|R|^{n-1}{}_{x}$ ".

·47 Notation when n = 2

In practice the results given in this thesis are concerned exclusively with the case $n = 2$. Denote by ϕ a vector in S^1 of the form $\phi = (\cos\theta, \sin\theta)$, so far in §·4 points in $\mathbb{R}^n$ and $\mathbb{R} \times S^{n-1}$ have been written $(r\phi)$ and (r,ϕ) respectively and these become $(r\cos\theta, r\sin\theta)$ and $(r,(\cos\theta, \sin\theta))$ in

$\mathbb{R}^2$ if one wishes to make explicit use of θ.

This constant repetition of cos and sin is dropped and instead the notation $(r,\theta) \in \mathbb{R}^2$ and $(r,\theta) \in \mathbb{R} \times S'$ will be used, explicit mention of the domains being made whenever there is the possibility of confusion.

To emphasise further the different domains, unless specifically stated otherwise, the letters f, p, F and P will be reserved for a function in the domain of $\mathcal{R}$, a function in the range of $\mathcal{R}$, a function in the domain of $\mathcal{P}$ and a function in the range of $\mathcal{P}$.

Under this notation the appearance of some equations will change, e.g.

The definition of Back Projection becomes:-

$$(\mathcal{B}t)(r,\theta) = \frac{1}{2} \int_0^{2\pi} t(r \cos(\theta - \psi), \psi) d\psi$$

The definition of the Radon transform is written

$$(\mathcal{R}f)(r,\theta) = \int_{L(r,\theta)} f$$

where $L(r,\theta)$ denotes the set previously written $\langle x, (\cos\theta, \sin\theta)\rangle = r$

·48 Continuity of $\mathcal{R}^{-1}$

·481 Lemma

$\mathcal{R}^{-1}$ is continuous iff there exists $m > 0$ such that

$$\left\|\frac{\|\mathcal{R}f\|}{\|f\|}\right\| \geq m > 0 \qquad \forall\, f \in \mathcal{L}^2(B^n) \setminus \{0\}.$$

Proof

This is a Corollary of (Taylor, 1958: Thoerems 3·1-A and 3·1-B).

·482 Theorem

The mapping $\mathcal{R}^{-1} : \mathcal{R}(\mathcal{L}^2(B \times S^{n-1}, \omega_2)) \to \mathcal{L}^2(B^n)$ is discontinuous.

Proof

The proof is by counter example for the case $n = 2$. (The notation of §·47 is used).

Let $\{f_k\} \subset C(B^2) \subset \mathcal{L}^2(B^2)$ be the sequence defined by

$$f_k(r,\phi) = \begin{cases} r^{2k} e^{i2k\phi} & |r| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $(\mathcal{R}f_k)(r,\phi) = \frac{2}{2k+1}\sqrt{1-r^2}\; U_{2k}(r)e^{i2k\phi}$

where $U_{2k}(r)$ is the Tchebichef polynomial of the second kind defined by

$$U_j(\cos\alpha) = \sin(j+1)\alpha / 1 \sin\alpha.$$

The form of $\mathcal{R}f_k$ is derived in (Marr, 1974).

It follows that

$$\|f_k\|^2 = \int_0^{2\pi} d\phi \int_0^1 r^{4k} r\,dr$$
$$= \frac{\pi}{2k+1}$$

and

$$\|\mathcal{R}f_k\|^2_{B^1 \times S^1, \omega_2} = \frac{4}{(2k+1)^2} \int_0^{2\pi} d\phi \int_{-1}^1 \sqrt{1-r^2}\; U_{2k}^2(r)\,dr$$
$$= \frac{4\pi^2}{(2k+1)^2}$$

Hence $$\frac{\|\mathcal{R}f_k\|_{B^1 \times S^1, \omega_2}}{\|f_k\|} = \left\{\frac{4\pi}{2k+1}\right\}^{\frac{1}{2}}$$

and given any m the requirement of §·481 is violated for all sufficiently large k.

·483 Corollary

The mappings $\mathcal{R}^{-1}:\mathcal{R}(\mathcal{L}^2(B^n,\omega_1)) \to \mathcal{L}^2(B^n)$

$$\mathcal{R}^{-1}:\mathcal{R}(\mathcal{L}^2(B^1 \times S^{n-1})) \to \mathcal{L}^2(B^n)$$

$$\mathcal{R}^{-1}:\mathcal{R}(\mathcal{L}^2(B^n)) \to \mathcal{L}^2(B^n)$$

are discontinuous.

Proof

By applying §·432 to the final equation in the proof of §·482.

Appendix 2·1: Relationship between continuous convolution $g = h * q$ and its discrete counterpart $\underline{g} = \underline{\underline{h}}^{*}\underline{q}$

Suppose $g = h * q$ i.e. that $g(r) = \int_{-\infty}^{\infty} h(r - r')q(r')dr'$ $\quad\cdot 01$

where $g = g^{*} + k_g$, $q = q^{*} + k_q$ (k_g and k_q constants) and g^{*}, q^{*} and h are zero for all sufficiently large $|r|$. Then

$$k_g = k_q \int_{-\infty}^{\infty} h(r)dr$$

and

$$g^{*} = h * q^{*} \qquad \cdot 02$$

In this appendix a discrete counterpart to the modified relationship E·02 will be derived. Various properties of $\underline{h}^{*}$ and its D.F.T. $\underline{H}^{*}$ will then be given. After this the discrete version of E·02 will be extended to provide a counterpart to E·01.

·1 Discrete version of $g^{*} = h * q^{*}$

Changing from the continuous equation E·02 to its discrete counterpart gives

$$g^{*}(r_j) = \Delta r \sum_{k=-\infty}^{\infty} h(r_j - r_k)q^{*}(r_k) \qquad \cdot 1$$

where $r_j = \left(j - \frac{N_2-1}{2}\right)\Delta r$ and N_2 has yet to be chosen (see §2·321). Writing $g_j^{*} = g^{*}(r_j)$, $q_j^{*} = q^{*}(r_j)$ and $h_j = h(j\Delta r)$, E·1 becomes

$$g_j^{*} = \Delta r \sum_{k=-\infty}^{\infty} h_{j-k}q_k^{*} \qquad \cdot 15$$

Note that the infinite summation converges since it contains only a finite number of non-zero terms. Define N', N" by

$$N' = \inf\left\{n \in \mathbb{N} \;\; |h(r) = 0 \;\; \forall\, |r| \geq \frac{n\Delta r}{2}\right\}$$

$$N'' = \inf\left\{n \in \mathbb{N} \;\; |q^{*}(r) = 0 \;\; \forall\, |r| \geq \frac{n\Delta r}{2}\right\}$$

Then

$$h_{j-k} \neq 0 \Rightarrow h((j-k)\Delta r) \neq 0 \Rightarrow |(j-k)\Delta r| < \frac{N'\Delta r}{2}$$

$$\Rightarrow -N' < 2(j-k) < N'$$

$$\Rightarrow \begin{cases} -N' + 1 \le 2(j-k) \le N' - 1 & \text{for } N' \text{ odd} \\ -N' + 2 \le 2(j-k) \le N' - 2 & \text{for } N' \text{ even} \end{cases}$$

$$\Rightarrow \begin{cases} 2j - N' + 1 \le 2k \le 2j + N' - 1 & \text{for } N' \text{ odd} \\ 2j - N' + 2 \le 2k \le 2j + N' - 2 & \text{for } N' \text{ even} \end{cases}$$

whence

$$j \le -1 \text{ and } h_{j-k} \neq 0 \Rightarrow \begin{cases} 2k \le N' - 3 & \text{for } N' \text{ odd} \\ 2k \le N' - 4 & \text{for } N' \text{ even} \end{cases} \qquad \cdot 2$$

and

$$j \ge N_2 \text{ and } h_{j-k} \neq 0 \Rightarrow \begin{cases} 2k \ge 2N_2 - N' + 1 \text{ for } N' \text{ odd} \\ 2k \ge 2n_2 - N' + 2 \text{ for } N' \text{ even} \end{cases} \qquad \cdot 3$$

Also

$$q_k^* = 0 \quad \text{if } |r_k| = \left| \left(k - \frac{N_2 - 1}{2} \right) \Delta r \right| \ge \frac{N''\Delta r}{2}$$

i.e. if $2k \le N_2 - N'' - 1$ or $2k \ge N_2 + N'' - 1$;

these two inequalities are equivalent to

$$\left.\begin{array}{l} 2k \le N_2 - N'' - 2 \text{ and } 2k \ge N_2 + N'' \text{ if } N_2 \text{ and } N'' \text{ are both even or both odd} \\ 2k \le N_2 - N'' - 1 \text{ and } 2k \ge N_2 + N'' - 1 \text{ if just one of } N_2 \text{ and } N'' \text{ is even} \end{array}\right\} \cdot 4$$

Now consider the case N_2, N' and N'' all even,

$$j \le -1 \text{ and } h_{j-k} \neq 0 \Rightarrow 2k \le N' - 4 \qquad \text{from E}\cdot 2$$

$$\Rightarrow 2k \le N_2 - N'' - 2$$

provided that $N' - 4 \le N_2 - N'' - 2$ or equivalently $N_2 \ge N' + N'' - 2$. Hence

$$j < 0 \text{ and } h_{j-k} \neq 0 \Rightarrow q_k^* = 0 \qquad \text{from E}\cdot 4$$

provided $N_2 \ge N' + N'' - 2$. Likewise

$$j \geq N_2 \text{ and } h_{j-k} \neq 0 \Rightarrow 2k \geq 2N_2 - N' + 2 \qquad \text{from E·3}$$

$$\Rightarrow 2k \geq N_2 + N''$$

provided that $2N_2 - N' + 2 \geq N_2 + N''$ or equivalently $N_2 \geq N' + N'' - 2$. Hence

$$j > N_2 - 1 \text{ and } h_{j-k} \neq 0 \Rightarrow q_k^* = 0 \qquad \text{from E·4}$$

provided $N_2 \geq N' + N'' - 2$.

Parallel reasoning for the other possible cases of N_2, N' and N'' even or odd shows that $N_2 \geq N' + N'' - 1$ is a sufficient condition for the implication

$$(j < 0 \text{ or } j > N_2 - 1) \text{ and } h_{j-k} \neq 0 \Rightarrow q_k^* = 0$$

to be true for all k.

Thus if $j < 0$ or $j > N_2 - 1$ then

$$h_{j-k} q_k^* = 0 \quad \forall k \qquad \cdot 45$$

and $g_j^* = \Delta r \sum_k h_{j-k} q_k^* = 0$ if $j < 0$ or $j > N_2 - 1$

or equivalently

$$0 \leq j \leq N_2 - 1 \Rightarrow g_{j+\ell N_2}^* = 0 \text{ unless } \ell = 0 \qquad \cdot 5$$

From E·15 and E·5, if $0 \leq j \leq N_2 - 1$ and $N_2 \geq N' + N'' - 1$ then

$$\begin{aligned} g_j^* &= \sum_{\ell=-\infty}^{\infty} g_{j+\ell N_2}^* \\ &= \sum_{\ell=-\infty}^{\infty} \Delta r \sum_{k=-\infty}^{\infty} h_{j+\ell N_2 - k}\, q_k^* \\ &= \sum_{k=-\infty}^{\infty} \left\{ \Delta r \sum_{\ell=-\infty}^{\infty} h_{j+\ell N_2 - k} \right\} q_k^* \\ &= \sum_{k=0}^{N_2 - 1} h_{jk}^* q_k^* \quad \text{where } h_{jk}^* = \Delta r \sum_{\ell=-\infty}^{\infty} h_{j+\ell N_2 - k} \end{aligned} \qquad \cdot 6$$

since $q_k^* = 0$ if $k < 0$ or $k > N_2 - 1$.

The order of summation may be changed for the following reason. By definition of h_j and q_j^*, for any ℓ there are only a finite number of non-zero terms $h_{j+\ell N_2-k}\, q_k^*$ (considered as functions of k), and there are only a finite number of values of ℓ for which this product is non-zero for any k. Thus the product is only non-zero for a finite number of ℓ and k. The summation is therefore absolutely convergent and the order of summation can be reversed.

Define the N_2-vectors $\underline{g}^*$ and $\underline{q}^*$ and the $N_2 \times N_2$ matrix $\underline{\underline{h}}^*$ by

$$(\underline{g}^*)_j = g_j^* \text{ , } (\underline{q}^*)_j = q_j^* \text{ and } (\underline{\underline{h}}^*)_{jk} = h_{jk}^*$$

then E·6 may be written as

$$\underline{g}^* = \underline{\underline{h}}^* \underline{q}^* \qquad \cdot 65$$

and this is taken as the discrete version of $g^* = h * q^*$.

·2 Properties of $\underline{\underline{h}}^*$

Now consider the form of the matrix $\underline{\underline{h}}^*$. By definition $\underline{\underline{h}}^*$ is circulant if

$$(\underline{\underline{h}}^*)_{j,k} = (\underline{\underline{h}}^*)_{(j-k)|N_2,0}$$

where $0 \le j,\ k \le N_2 - 1$. From the definition of $\underline{\underline{h}}^*$;

$$(\underline{\underline{h}}^*)_{(j-k)|N_2,0} = \Delta r \sum_{\ell=-\infty}^{\infty} h_{(j-k)|N_2+\ell N_2 - 0}$$

$$= \Delta r \sum_{\ell=-\infty}^{\infty} h_{j-k+\ell N_2}$$

$$= (\underline{\underline{h}}^*)_{jk}$$

Thus $\underline{\underline{h}}^*$ is circulant and is completely determined by its first column $\underline{h}^*$ where

$$(\underline{h}^*)_j = (\underline{\underline{h}}^*)_{j0} = \Delta r \sum_{\ell=-\infty}^{\infty} h_{j+\ell N_2}$$

$$= \Delta r \, (h_j + h_{j-N_2}) \qquad 0 \le j \le N_2 - 1$$

or since $h_j = 0$ for $|j| \ge \frac{N_2}{2} > \frac{N'}{2}$

$$(\underline{h}^*)_j = \begin{cases} \Delta r \, h_j & 0 \le j \le N_2/2 \\ \Delta r \, h_{j-N_2} & N_2/2 < j \le N_2 - 1 \end{cases} \qquad \cdot 7$$

·3 Relation between $\underline{H}^* = \underline{\underline{W}}\,\underline{h}^*$

The discrete fourier transform of $\underline{h}^*$ is defined by $\underline{H}^* = \underline{\underline{W}}\,\underline{h}^*$. A relationship between this and the continuous fourier transform H of h is now derived.

$$(\underline{H}^*)_k = (\underline{\underline{W}}\,\underline{h}^*)_k$$

$$= \frac{1}{\sqrt{N_2}} \sum_{j=0}^{N_2-1} e^{-2\pi ijk/N_2} \, h_j^*$$

$$= \frac{\Delta r}{\sqrt{N_2}} \left\{ \sum_{j=0}^{N_2/2} e^{-2\pi ijk/N_2} \, h_j + \sum_{j=\frac{N_2}{2}+1}^{N_2-1} e^{-2\pi ijk/N_2} \, h_{j-N_2} \right\}$$

$$= \frac{\Delta r}{\sqrt{N_2}} \left\{ \sum_{j=0}^{N_2/2} e^{-2\pi ijk/N_2} \, h_j + \sum_{j'=-\frac{N_2}{2}+1}^{-1} e^{-2\pi i(j'+N_2)k/N_2} \, h_{j'} \right\}$$

$$= \frac{\Delta r}{\sqrt{N_2}} \left\{ \sum_{j=0}^{N_2/2} e^{-2\pi ijk/N_2} \, h_j + \sum_{j'=-\frac{N_2}{2}+1}^{-1} e^{-2\pi ij'/N_2} \, h_{j'} \right\}$$

$$= \frac{\Delta r}{\sqrt{N_2}} \sum_{j=-\frac{N_2}{2}+1}^{N_2/2} e^{-2\pi ijk/N_2} \, h_j$$

$$= \frac{\Delta r}{\sqrt{N_2}} \sum_{j=-\infty}^{\infty} e^{-2\pi ijk/N_2} h_j \quad \text{since} \quad h_j = 0 \quad \forall |j| \geq \frac{N_2}{2}$$

$$= \frac{1}{\sqrt{N_2}} \sum_{\ell=-\infty}^{\infty} H((k - \ell N_2)\Delta R) \quad \text{from Ap. 1·1·3 E·3}$$

$$\simeq \frac{1}{\sqrt{N_2}} \{H(k\Delta R) + H((k - N_2)\Delta R)\} \quad \text{provided } H(R) \simeq 0 \text{ for } |R| \geq N_2\Delta R/2 \qquad \cdot 8$$

·4 Discrete version of $g = h * q$

Define $\underline{g}$, $\underline{q}$, $\underline{k}_g$, $\underline{k}_q$ by:-

$$(\underline{g})_j = g(r_j) = (\underline{g}^*)_j + k_g$$

$$(\underline{q})_j = q(r_j) = (\underline{q}^*)_j + k_q$$

$$(\underline{k}_g)_j = k_g$$

$$(\underline{k}_q)_j = k_q$$

Then $(\underline{\underline{h}}^* \underline{k}_q)_j = k_q \sum_{k=0}^{N_2-1} h^*_{jk}$ and it is straightforward to show that

$$\sum_{k=0}^{N_2-1} h^*_{jk} = \Delta r \sum_{k=-\frac{N_2}{2}+1}^{N_2/2} h_k$$

$$= \Delta r \sum_{k=-\infty}^{\infty} h_k \quad \text{since} \quad h_k = 0 \; \forall |k| \geq N_2/2$$

$$\simeq \int_{-\infty}^{\infty} h(r)\,dr$$

Hence
$$(\underline{\underline{h}}^* \underline{k}_q)_j \simeq k_q \int_{-\infty}^{\infty} h(r)\,dr = k_g$$

and $\underline{\underline{h}}^* \underline{k}_q \simeq \underline{k}_g$ ·9

Adding E·9 into E·65 and using the definitions of $\underline{q}$ and $\underline{g}$

$$\underline{g} \simeq \underline{\underline{h}}^* \underline{q} \qquad \cdot 91$$

This equation is thus taken as the discrete counterpart of $g = h * q$.

·5 Summary

Given that $g = h * q$

then $(\underline{q})_j = q(r_j)$ and $(\underline{g})_j = g(r_j)$ are related by

$$\underline{g} = \underline{\underline{h}}^* \underline{q}$$

where $(\underline{\underline{h}}^*)_{jk} = h^*_{(j-k)|N_2}$ and $h^*_j = \begin{cases} \Delta r\, h_j & 0 \le j \le N_2/2 \\ \Delta r\, h_{j-N_2} & N_2/2 \le j \le N_2 - 1 \end{cases}$

The D.F.T. $\underline{H}^*$ of $\underline{h}^*$ defined by $\underline{H}^* = \underline{\underline{W}}\, \underline{h}^*$ is related to the C.F.T. H of h by

$$(\underline{H}^*)_k \simeq \frac{1}{\sqrt{N_2}} \{H(k\Delta R) + H((k - N_2)\Delta R)\}$$

provided $H(R) \simeq 0 \;\; \forall |R| \ge N_2 \Delta R/2$.

Appendix 2·2 Necessary and Sufficient Conditions for a solution to the Constrained Optimization Problem.

·1 Necessary and Sufficient conditions for existence of optimal solution to general non linear programming problem (N.L.P.)

Consider the problem "max $f(y)$ subject to $g(y) \geq 0 \quad y \in \mathbb{R}^n$". The following series of definitions and Theorems is taken from W.I. Zangwill 1969. [ ] denotes a reference to this work.

Defn ·01 Given the constraint $g(y) \geq 0$ the feasible set F for the N.L.P. is

$$F = \{y \mid g(y) \geq 0\} \qquad [p4]$$

Defn ·02 If $y^* \in F$ then y^* is an optimal solution to the N.L.P. iff

$$f(y^*) \geq f(y) \quad \forall\ y \in F \qquad [p3]$$

Defn ·03 For all $y \in F$ the sets $D(y)$ and $\mathcal{D}(y)$ are defined by

$$D_g(y) = \{d \mid \exists \sigma > 0 \text{ such that } \tau \in [0,\sigma] \Rightarrow y + \tau d \in F\} \qquad [p37]$$

$$\mathcal{D}_g(y) = \{d \mid \nabla g(y)'d \geq 0\} \qquad [p39]$$

Defn ·04 For the constraint $g(y) \geq 0$ the constraint qualification is said to hold if

$$\mathcal{D}(y^*) \subset \overline{D}(y^*) \qquad [p40]$$

Theorem ·1 (Kuhn-Tucker)

Consider the N.L.P. If f,g are differentiable and concave and the constraint qualification holds then:-

y^* is optimal $\Rightarrow$ 1) $y^* \in F$ ·1

$\exists \lambda \geq c$ such that

2) $\lambda g(y^*) = 0$ ·2

3) $\nabla f(y^*) + \lambda \nabla g(y^*) = 0$ ·3

[p40 Th 2·14]

Theorem ·2

Consider the N.L.P. where f, g are differentiable and concave. If y^* satisfies E·1, E·2, E·3 then y^* is optimal.

[p 43 Th 2·15
p 54 Ex 2·8(a)(b)]

Theorem ·3

If a function f is twice continuously differentiable on a convex set and the Hessian matrix

$$\frac{\partial^2 f}{\partial y_j \, \partial y_k}$$

is -ve semi-definite then f is concave. [p30 L 2·5]

These theorems may be gathered together to give:-

Theorem ·4

If g is such that the constraint qualification holds and g,f are both twice continuously differentiable on $\mathbb{R}^n$ with -ve semi-definite Hessians then:-

y^* is optimal $\Leftrightarrow$ 1) $y^* \in F$

$\exists\, \lambda \geq 0$ such that

2) $\lambda g(y^*) = 0$

3) $\nabla f(y^*) + \lambda \nabla g(y^*) = 0$

·2 Theorems concerning the constraint qualification

Theorem ·45

If $g(y) = 1 - y'SSy$ where S is a real diagonal matrix then the constraint qualification holds.

Proof

$$d \in \mathcal{D}(y) \Rightarrow \nabla g(y)'d \geq 0$$

but $g(y) = 1 - y'S'Sy \Rightarrow \nabla g(y) = -2S'Sy$

$\therefore \nabla g(y)'d \geq 0 \Rightarrow -2y'S'Sd \geq 0$

$\therefore d \in \mathcal{D}(y) \Rightarrow y'S'Sd \leq 0$.

(1) if $y'S'Sd < 0$

lemma : $y'S'Sd < 0 \Rightarrow Sd \neq 0 \Rightarrow d'S'Sd \neq 0$.

hence for any y,d such that $y'S'Sd < 0$

$$\tau^2 d'S'Sd + 2\tau y'S'Sd = \tau d'S'Sd \left\{\tau + 2\,\frac{y'S'S'd}{d'S'Sd}\right\} \text{ since } d'S'Sd \neq 0$$

$$\leq 0 \quad \forall\, \tau \in \left[0, \left|2\,\frac{y'S'Sd}{d'S'Sd}\right|\right] \text{ since } y'S'Sd < 0$$

further, if $y \in F$.

$$\tau^2 d'S'Sd + 2\, y'S'Sd \leq 0$$

$\Rightarrow \quad \tau^2 d'S'Sd + 2\tau y'S'Sd + y'S'S_y \leq y'S'Sy \leq 1$ since $y \in F \Rightarrow y'S'S_y \leq 1$

$\Rightarrow \quad (y + \tau d)'S'S(y + \tau d) \leq 1$

$\Rightarrow \quad y + \tau d \in F \qquad \forall\, \tau \in \left[0, \left|2\,\frac{y'S'Sd}{d'S'Sd}\right|\right]$

hence $y'S'Sd < 0 \Rightarrow d \in D(y)$

$\Rightarrow d \in \overline{D}(y)$ ·4

(2) if $y'S'Sd = 0$

$$1 - (y + \tau d)'S'S(y + \tau d)$$

$$= 1 - y'S'Sy - 2\tau y'S'Sd - \tau^2 d'S'Sd$$

if $Sd = 0$ $\quad 1 - (y + \tau d)'S'S(y + \tau d)$

$$= 1 - y'S'Sy \geq 0 \quad \text{since } y \in F$$

$\therefore y + \tau d \in F$

$\therefore d \in \overline{D}(y)$ for any σ

if $Sy = 0 \quad 1 - (y + \tau d)'S'S(y + \tau d)$

$$= 1 - \tau^2 d'S'Sd$$

$$\geq 0 \quad \text{if} \quad \tau \in \left[0, \frac{1}{\sqrt{d'S'Sd}}\right]$$

$\therefore\ d \in \overline{D}(y)$ by putting $\sigma = \dfrac{1}{\sqrt{d'S'Sd}}$ in Defn ·03

if $Sd \neq 0$ and $Sy \neq 0$ then define $d_k = d - \frac{1}{k} y$

then $$y'S'Sd_k = y'S'Sd - \frac{1}{k} y'S'Sy$$

$$= -\frac{1}{k} y'S'Sy$$

$$< 0 \quad \forall k > 0$$

and from E·4 $y'S'Sd_k < 0 \Rightarrow d_k \in D(y)$. Further $\|d_k - d\| \to 0$ as $k \to \infty$ so that d is a limit point of $D(y)$. Hence $d \in \overline{D}(y)$.

Thus $d \in \mathscr{D}(y) \Rightarrow y'S'Sd \leq 0 \Rightarrow d \in \overline{D}(y) \quad \forall y \in F$ and the result

$\mathscr{D}(y) \subset \overline{D}(y)$ is proved.

Theorem ·5

Let $g(y)$ be such that $\mathscr{D}_g(y) = \overline{D}_g(y)$. If $f(y) = g(a - y)$ then

$$\mathscr{D}_f(y) \subset \overline{D}_f(y).$$

Proof

$$[\mathscr{D}_g(y) \subset D_g(y)] \Leftrightarrow [\nabla g(y)'d \geq 0 \Rightarrow (\exists \sigma > 0 \text{ s.t. } \tau \in [0,\sigma] \Rightarrow g(y + \tau d) \geq 0]$$

$$\begin{aligned}
\nabla f(y)'d \geq 0 \quad &\Rightarrow \quad -\nabla g(a - y)'d \geq 0 \\
&\Rightarrow \quad \nabla g(a - y)'(-d) \geq 0 \\
&\Rightarrow [\exists \sigma > 0 \text{ s.t. } \tau \in [0,\sigma] \Rightarrow g((a - y) + \tau(-d)) \geq 0] \\
&\Rightarrow [\exists \sigma > 0 \text{ s.t. } \tau \in [0,\sigma] \Rightarrow g(a - (y + \tau d)) \geq 0] \\
&\Rightarrow [\exists \sigma > 0 \text{ s.t. } \tau \in [0,\sigma] \Rightarrow f(y + \tau d) \geq 0]
\end{aligned}$$

Hence $\mathscr{D}_f(y) \subset D_f(y) \subset \overline{D}_f(y)$.

Theorem ·6

Let $g(y)$ be such that $\mathscr{D}_g(y) \subset \overline{D}_g(y)$. If $f(y) = c\, g(y)$ where $c > 0$ then

$$\mathscr{D}_f(y) \subset \overline{D}_f(y)$$

Proof

$$\begin{aligned}\nabla f(y)'\, d \ge 0 &\Rightarrow \quad c \;\; \nabla g(y)'\, d \ge 0 \\ &\Rightarrow \nabla g(y)'\, d \ge 0 \\ &\Rightarrow [\exists\, \sigma > 0 \;\; \text{s.t.}\; \tau \in [0,\sigma] \Rightarrow g(y + \tau d) \ge 0] \\ &\Rightarrow [\exists\, \sigma > 0 \;\; \text{s.t.}\; \tau \in [0,\sigma] \Rightarrow f(y + \tau d) \ge 0] \qquad \text{since } c > 0\end{aligned}$$

Hence $\mathscr{D}_f(y) \subset D_f(y) \subset \overline{D}_f(y)$

Theorem ·7

If $g(y) = c - (a - y)'T(a - y)$ where $c > 0$ and T is real and diagonal then the constraint qualification holds.

Proof

This follows immediately from Th ·45, Th ·5, Th ·6.

·3 Necessary and Sufficient conditions for the existence of an optimal solution to "$\min\{N(\underline{\underline{C}}\, \hat{\underline{Q}})^T(\underline{\underline{C}}\, \hat{\underline{Q}})\}$ subject to $(\hat{\underline{G}} - \sqrt{N}\, \underline{\underline{H}}\, \hat{\underline{Q}})^T(\hat{\underline{G}} - \sqrt{N}\, \underline{\underline{H}}\, \hat{\underline{Q}}) \le e^2$"

Consider the problem as

$$\max\, \{-N\, \hat{\underline{Q}}^T \underline{\underline{C}}^T \underline{\underline{C}}\, \hat{\underline{Q}}\} \;\; \text{subject to} \;\; e^2 - \; (\hat{\underline{G}} - \sqrt{N}\, \underline{\underline{H}}\, \hat{\underline{Q}})^T(\hat{\underline{G}} - \sqrt{N}\, \underline{\underline{H}}\, \hat{\underline{Q}}) \ge 0$$

Define a vector $\underline{y}$ with 2N components by

$$(\underline{y})_j = \begin{cases} \operatorname{Re}\, \{(\hat{\underline{Q}})_j\} & j = 0(1)N-1 \\ \operatorname{Im}\, \{(\hat{\underline{Q}})_{j-N}\} & j = N(1)2N-1 \end{cases}$$

Define $\underline{G}_1$ by $(\underline{G}_1)_j = \begin{cases} \dfrac{1}{\sqrt{N}\, H_{jj}} \hat{G}_j & \text{if } H_{jj} \neq 0 \\ 0 & \text{if } H_{jj} = 0 \end{cases}$

and a vector $\underline{G}_2$ by

$$\underline{G}_2 = \hat{\underline{G}} - \sqrt{N}\, \underline{\underline{H}}\, \underline{G}_1$$

so that

$$\underline{\underline{H}}\, \underline{G}_2 = \underline{\underline{H}}^T \underline{G}_2 = (\underline{G}_2^T \underline{\underline{H}})^T = \underline{O}$$

then

$$e^2 - (\hat{\underline{G}} - \sqrt{N}\, \underline{\underline{H}}\, \hat{\underline{Q}})^T (\hat{\underline{G}} - \sqrt{N}\, \underline{\underline{H}}\, \hat{\underline{Q}})$$

$$= (e^2 - \underline{G}_2^T \underline{G}_2) - (\underline{G}_1 - \hat{\underline{Q}})^T (\sqrt{N}\, \underline{\underline{H}})^T (\sqrt{N}\, \underline{\underline{H}}) (\underline{G}_1 - \hat{\underline{Q}})$$

Since $\underline{\underline{C}}$ and $\sqrt{N}\, \underline{\underline{H}}$ are diagonal, $\underline{\underline{C}}^T \underline{\underline{C}}$ and $(\sqrt{N}\, \underline{\underline{H}})^T (\sqrt{N}\, \underline{\underline{H}})$ are real positive semi-definite and diagonal. Brief consideration of the quadratic forms $\hat{\underline{Q}}^T \underline{\underline{C}}^T \underline{\underline{C}}\, \hat{\underline{Q}}$ and $(\underline{G}_1 - \hat{\underline{Q}})^T (\sqrt{N}\, \underline{\underline{H}})^T (\sqrt{N}\, \underline{\underline{H}}) (\underline{G}_1 - \hat{\underline{Q}})$ shows that they are real quadratic forms in the 2N components of $\underline{y}$, and further, that all cross products of the form $y_j y_k (j \neq k)$ are absent.

Thus

$$N\, \hat{\underline{Q}}^T \underline{\underline{C}}^T \underline{\underline{C}}\, \underline{Q}$$

may be written in the form

$$\underline{y}' \underline{\underline{P}}\, \underline{y}$$

and

$$(e^2 - \underline{G}_2^T \underline{G}_2) - (\underline{G}_1 - \hat{\underline{Q}})^T (\sqrt{N}\, \underline{\underline{H}})^T (\sqrt{N}\, \underline{\underline{H}}) (\underline{G}_1 - \hat{\underline{Q}})$$

may be written in the form

$$c - (\underline{a} - \underline{y})'\, \underline{\underline{T}} (\underline{a} - \underline{y})$$

where $\underline{\underline{P}}$, $\underline{\underline{T}}$ are real diagonal, + ve semi-definite matrices. Thus with

$$g(y) = (e^2 - \underline{G}_2^T \underline{G}_2) - (\underline{G}_1 - \hat{\underline{Q}})^T (\sqrt{N}\, \underline{\underline{H}})^T (\sqrt{N}\, \underline{\underline{H}}) (\underline{G}_1 - \hat{\underline{Q}})$$

$$= c - (\underline{a} - \underline{y})'\underline{\underline{T}}(\underline{a} - \underline{y})$$

and

$$f(y) = -N\,\hat{\underline{Q}}^T\underline{\underline{C}}^T\underline{\underline{C}}\,\hat{\underline{Q}}$$

$$= -\,\underline{y}'\underline{\underline{P}}\,\underline{y}$$

it follows from Th·7 that the constraint qualification holds for g (it is assumed that $e^2 - \underline{G}_2^T\underline{G}_2 \geq 0$ else $F = \phi$ and there is no optimization problem). It is also clear that the Hessians for g and f exist and are -T and -P respectively, and hence are negative semi-definite. Thus applying Th·4 gives

Theorem ·8

In the problem $\min\{N\,\underline{Q}^T\underline{\underline{C}}^T\underline{\underline{C}}\,\underline{Q}\}$ subject to $(\hat{\underline{G}} - \sqrt{N}\,\underline{\underline{H}}\,\hat{\underline{Q}})^T(\hat{G} - \sqrt{N}\,\underline{\underline{H}}\,\hat{\underline{Q}}) \leq e^2$ $\hat{\underline{Q}}$ is optimal iff

1) $(\hat{\underline{G}} - \sqrt{N}\,\underline{\underline{H}}\,\hat{\underline{Q}})^T(\hat{\underline{G}} - \sqrt{N}\,\underline{\underline{H}}\,\hat{\underline{Q}}) \leq e^2$

$\exists\lambda \leq 0$ such that

2) $\lambda[e^2 - (\hat{\underline{G}} - \sqrt{N}\,\underline{\underline{H}}\,\hat{\underline{Q}})^T(\hat{\underline{G}} - \sqrt{N}\,\underline{\underline{H}}\,\hat{\underline{Q}})] = 0$

3) $\nabla[N\,\hat{\underline{Q}}^T\underline{\underline{C}}^T\underline{\underline{C}}\,\hat{\underline{Q}} + \lambda\,\{e^2 - (\hat{\underline{G}} - \sqrt{N}\,\underline{\underline{H}}\,\hat{\underline{Q}})^T(\hat{\underline{G}} - \sqrt{N}\,\underline{\underline{H}}\,\hat{\underline{Q}})\}] = 0$

where ∇ is interpreted as the operator

$$\left(\frac{\partial}{\partial\,\mathrm{re}(\underline{Q})_0}, \ldots, \frac{\partial}{\partial\,\mathrm{Re}(\underline{Q})_{N-1}}, \frac{\partial}{\partial\,\mathrm{Im}(\underline{Q})_0}, \ldots, \frac{\partial}{\partial\,\mathrm{Im}(\underline{Q})_{N-1}}\right) .$$

Appendix 2·3 Conditions for non-singularity of $(\underline{\underline{H}}^T\underline{\underline{H}} + \gamma\underline{\underline{C}}^T\underline{\underline{C}})$

Note first that, since $\underline{\underline{H}}$ and $\underline{\underline{C}}$ are diagonal $\underline{\underline{H}}^T\underline{\underline{H}}$ and $\underline{\underline{C}}^T\underline{\underline{C}}$ are diagonal

and $(\underline{\underline{H}}^T\underline{\underline{H}})_{jj} \geq 0 \qquad j = 0(1)N-1$

and $(\underline{\underline{C}}^T\underline{\underline{C}})_{jj} \geq 0 \qquad j = 0(1)N-1$

and $\gamma > 0$ since $\lambda > 0$.

Theorem ·1

$$(\underline{\underline{C}}^T\underline{\underline{C}})_{jj} = \frac{16}{N}\sin^4\left(\frac{\pi j}{N}\right) \qquad j = 0(1)N-1$$

Proof

By definition $\underline{\underline{C}}$ has diagonal elements $\underline{C} = \underline{\underline{W}}\,\underline{c}$. Hence

$$(\underline{\underline{C}})_{jj} = (\underline{\underline{W}}\,\underline{c})_j = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{-2\pi ijk/N}c_k$$

$$= \frac{1}{\sqrt{N}}\{-2 + e^{-2\pi ij/N} + e^{-2\pi ij(N-1)/N}\} \text{ since } \underline{c} = (-2,1,0,\ldots,0,1)'$$

$$= \frac{-4}{\sqrt{N}}\sin^2\left(\frac{\pi j}{N}\right)$$

$$\therefore (\underline{\underline{C}}^T\underline{\underline{C}})_{jj} = \overline{(\underline{\underline{C}})_{jj}}(\underline{\underline{C}})_{jj} = \frac{16}{N}\sin^4\left(\frac{\pi j}{N}\right) \qquad j = 0(1)N-1 .$$

Thus $(\underline{\underline{C}}^T\underline{\underline{C}})_{jj} > 0$ for $j = 1(1)N-1$ and $(\underline{\underline{C}}^T\underline{\underline{C}})_{OO} = 0$, hence

$$(\underline{\underline{H}}^T\underline{\underline{H}} + \gamma\underline{\underline{C}}^T\underline{\underline{C}})_{jj} > 0 \qquad j = 1(1)N-1$$

and $(\underline{\underline{H}}^T\underline{\underline{H}} + \gamma\underline{\underline{C}}^T\underline{\underline{C}})$ is singular iff $(\underline{\underline{H}}^T\underline{\underline{H}})_{OO} = 0$ iff $(\underline{\underline{H}})_{OO} = 0$ iff $(\underline{H})_O = 0$

But $(\underline{H})_O = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} h_j$

So $\underline{\underline{H}}^T\underline{\underline{H}} + \gamma\underline{\underline{C}}^T\underline{\underline{C}}$ is non-singular if and only if $\Sigma h_j = 0$. However, for the function h considered here (see E·2) $h_j \geq 0$ for all j and $\underline{h} \neq 0$, hence $\underline{\underline{H}}^T\underline{\underline{H}} + \gamma\underline{\underline{C}}^T\underline{\underline{C}}$ is non-singular.

Appendix 2·4: Discrete version of $\mathcal{F}_1^{-1}\left\{\dfrac{\lambda_c|R|}{|R|^n + \lambda_c} \quad \mathcal{F}_1\hat{p}\right\}$

Define $s = \mathcal{F}_1^{-1}\left\{\dfrac{\lambda_c|R|}{|R|^n + \lambda_c}\right\}$, a discrete counterpart of the function $t = s * \hat{p}$ is developed.

From Appendix 2·1 the discrete version of $g^* = h * q^*$ is given by

$$\underline{g}^* = \underline{\underline{h}}^* \underline{q}^* \quad \text{where} \quad (\underline{g}^*)_j = g^*(r_j) \qquad r_j = \left\{j - \frac{N_2-1}{2}\right\} \Delta r$$

$$(\underline{q}^*)_j = q^*(r_j)$$

and $\underline{\underline{h}}^*$ is related to $\underline{\underline{H}}^*$ and $\underline{H}^*$ in the usual way (see Appendix 1·1·15) and

$$(\underline{H}^*)_k = \frac{1}{\sqrt{N_2}} \{H(k\Delta R) + H((k - N_2)\Delta R)\}$$

provided $H(R) \simeq 0 \;\forall\; |R| \geq N_2\Delta R/2$.

This result is now reapplied to the relation $t = s * \hat{p}$ giving

$$\underline{t} = \underline{\underline{s}}^* \hat{\underline{p}} \quad \text{where} \quad (\underline{t})_j = t(r_j)$$

$$(\hat{\underline{p}})_j = \hat{p}(r_j)$$

$$r_j = \left\{j - \frac{N_3 - 1}{2}\right\} \Delta r$$

and $\underline{\underline{s}}^*$ is related to $\underline{\underline{S}}^*$ and $\underline{S}^*$ as usual

$$(\underline{S}^*)_k = \frac{1}{\sqrt{N_3}} \{S(k\Delta R) + S((k - N_3)\Delta R)\}$$

provided $S(R) \simeq 0 \;\forall |R| \geq N_3\Delta R/2$.

It follows immediately from the definition of s that $S(R) = \lambda_c|R|/(|R|^n + \lambda_c)$.
This result is now manipulated to yield an expression for $\underline{t}$.

$$S(k\Delta R) = \frac{\lambda_c(k\Delta R)}{(k\Delta R)^n + \lambda_c} = \Delta R \;\frac{\lambda_d k}{k^n + \lambda_d}$$

where $\lambda_d = \lambda_c \Delta R^{-n}$. Similarly

$$S((k - N_3)\Delta R) = S((N_3 - k)\Delta R)$$

$$= \Delta R \frac{\lambda_d (N_3 - k)}{(N_3 - k)^n + \lambda_d}$$

so that

$$\underline{\underline{s}}^* = \frac{1}{\sqrt{N_3}} \Delta R \ \lambda_d \ \underline{\underline{\Lambda}} \ (\underline{\underline{\Lambda}}^n + \lambda_d \underline{\underline{I}})^{-1}$$

where $\underline{\underline{\Lambda}} = \text{diag}(\underline{\Lambda})$

and $$(\underline{\Lambda})_k = \begin{cases} k & 0 \le k \le N_3/2 \\ N_3 - k & N_3/2 < k \le N_3 - 1. \end{cases}$$

This may be substituted into the expression for $\underline{t}$ as follows:-

$$\underline{t} = \underline{\underline{s}}^* \hat{\underline{p}}$$

$$= \underline{\underline{W}}^{-1} (\underline{\underline{W}} \ \underline{\underline{s}}^* \underline{\underline{W}}^{-1}) \ \underline{\underline{W}} \ \hat{\underline{p}}$$

$$= \sqrt{N_3} \ \underline{\underline{W}}^{-1} \ \underline{\underline{S}}^* \ \underline{\underline{W}} \ \hat{\underline{p}} \quad \text{from Appendix 1.1.21}$$

$$= \Delta R \ \underline{\underline{W}}^{-1} \ \lambda_d \ \underline{\underline{\Lambda}} \ (\underline{\underline{\Lambda}}^n + \lambda_d \underline{\underline{I}})^{-1} \ \underline{\underline{W}} \ \hat{\underline{p}}$$

$$= (N_3 \ \Delta r)^{-1} \ \ \underline{\underline{W}}^{-1} \ \lambda_d \ \underline{\underline{\Lambda}} \ (\underline{\underline{\Lambda}}^n + \lambda_d \underline{\underline{I}})^{-1} \ \underline{\underline{W}} \ \hat{\underline{p}}$$

and this is the desired result.

Three points have been glossed over in obtaining this result:-

1) Because the value $N = N_2$ was appropriate for $\underline{g}^* = \underline{\underline{h}}^* \underline{g}^*$ it does not follow that the same value will be appropriate for calculating $\underline{t} = \underline{\underline{s}}^* \hat{\underline{p}}$. This fact is explicitly acknowledged by using the value $N = N_3$. This immediately raises two further points.

2) Clearly the sample spacing Δr for $\hat{\underline{p}}$ is the same as that for $\underline{g}^*$ (or more precisely $\hat{\underline{g}}$), $\hat{\underline{g}}$ and $\underline{m}$ and since $N \Delta r \Delta R = 1$ (see §1·1·3 E·7) the change from $N = N_2$ to $N = N_3$ implies that ΔR changes from $(N_2 \Delta r)^{-1}$ to $(N_3 \Delta r)^{-1}$ so the ΔR used in connection with the equation $\underline{g}^* = \underline{\underline{h}}^* \underline{g}^*$ is different to that used with $\underline{t} = \underline{\underline{s}}^* \hat{\underline{p}}$. Nevertheless the abuse of notation is left rather than introduce yet another symbol.

3) The function p is estimated as the N_2-vector $\hat{\underline{p}}$. If $\underline{t}$ is calculated using the N_3-vector $\hat{\underline{p}}$, where have the extra data values come from? (When considered further in §2·545 it will be seen that $N_3 \geq N_2$). It is known that $\hat{p}(r) = 0$ for all sufficiently large $|r|$ and (see §3) various parameters will be chosen so that r_0 and r_{N_2-1} both lie in the region where $\hat{p}(r) = 0$ (by taking r_0 sufficiently negative and r_{N_2-1} sufficiently positive). The vector $\hat{\underline{p}}$ is therefore padded out by adding zero components as necessary. Formally, denote the N_3-vector by $\hat{\underline{p}}^a$ (a for augmented), then

$$(\hat{\underline{p}}^a)_j = \begin{cases} (\hat{\underline{p}})_{\left(j - \frac{N_3-N_2}{2}\right)} & \frac{N_3 - N_2}{2} \leq j \leq \frac{N_3 + N_2}{2} - 1 \\ 0 & 0 \leq j < \frac{N_3 - N_2}{2} \text{ or } \frac{N_3 + N_2}{2} \leq j \leq N_3 - 1 \end{cases}$$

And once again notation has been abused, as it is the vector $\hat{\underline{p}}^a$ which is used in the discussion of the equation $\underline{t} = \underline{\underline{s}}^* \hat{p}$.

Appendix 2·5 : Interpolation

The quantities $[r]$ and $]r[$ are defined as the integer part of r and the fractional part of r respectively. Formally

$$[r] = \sup(\{x|x \le r\} \cap \mathbb{N})$$

$$]r[= r - [r]$$

Now defined $\underline{v}_1$ and $\underline{v}_2$ by

$$(\underline{v}_1(r))_k = \delta_{k,[r+\frac{1}{2}]}$$

$$(\underline{v}_2(r))_k = (1 -]r[)\ \delta_{k,[r]} +]r[\ \delta_{k,[r+1]}$$

Let $f(r)$ be a function defined on $\{1,2,\ldots,N\}$ and $\underline{f}$ be defined by

$$(\underline{f})_k = f(k)$$

then

$$\underline{v}_1(r)'\underline{f} \quad \text{and} \quad \underline{v}_2(r)'\underline{f}$$

are the step function and linear interpolated values of $\underline{f}$ at the point r respectively.

Appendix 2·6 : Evaluation of $\int_0^\infty \frac{\zeta}{\zeta^5 + 1}\, d\zeta$

Denote by c_1, c_2, s_1, s_2 the quantities $\cos\frac{2\pi}{5}$, $\cos\frac{4\pi}{5}$, $\sin\frac{2\pi}{5}$ and $\sin\frac{4\pi}{5}$ respectively.

$$\int_0^\infty \frac{\zeta}{\zeta^5+1}\,d\zeta = \int_0^\infty -\frac{1}{5}\left\{\frac{1}{1+\zeta} + c_2\,\frac{2\zeta + 2c_1}{\zeta^2 + 2c_1\zeta + 1} + \frac{2c_1(1-c_2)}{(\zeta+c_1)^2 + s_1^2}\right.$$

$$\left. + c_1\,\frac{2\zeta + 2c_2}{\zeta^2 + 2c_2\zeta + 1} + \frac{2c_2(1-c_1)}{(\zeta+c_2)^2 + s_2^2}\right\}$$

$$= -\frac{1}{5}\left\{\ln(1+\zeta) + c_2\ln(1 + 2c_1\zeta + \zeta^2) + 2c_1\,\frac{1-c_2}{s_1}\tan^{-1}\left(\frac{\zeta+c_1}{s_1}\right)\right.$$

$$\left.\left. + c_1\ln(1 + 2c_2\zeta + \zeta^2) + 2c_2\,\frac{1-c_1}{s_2}\tan^{-1}\left(\frac{\zeta+c_2}{s_2}\right)\right\}\right|_0^\infty$$

$$= f(\zeta)\Big|_0^\infty \qquad \text{(where this equation defines } f\text{)}$$

$$f(0) = -\frac{1}{5}\left\{\begin{array}{l} 0 + 0 + 2c_1 \cdot 2s_1\tan^{-1}(c_1/s_1) \\ \quad + 0 + 2c_2 \cdot 2s_2\tan^{-1}(c_2/s_2)\end{array}\right\}$$

$$= -\frac{1}{5}\left\{4c_1s_1\left(\frac{\pi}{2} - \frac{2\pi}{5}\right) - 4c_2s_2\left(\frac{\pi}{2} - \frac{\pi}{5}\right)\right\}$$

$$= \frac{-4\pi}{50}\,(c_1s_1 - 3c_2s_2)$$

$$\lim_{\zeta\to\infty} f(\zeta) = -\frac{1}{5}\lim\left\{\begin{array}{l} \ln\zeta + \ln(\zeta^{-1} + 1) \\ + 2c_2\ln\zeta + c_2\ln(\zeta^{-2} + 2c_1\zeta^{-1} + 1) \\ + 2c_1\ln\zeta + c_1\ln(\zeta^{-2} + 2c_2\zeta^{-1} + 1) \\ + 4c_1s_1\tan^{-1}[(\zeta + c_1)/s_1] \\ + 4c_2s_2\tan^{-1}[(\zeta + c_2)/s_2]\end{array}\right\}$$

and since $1 + 2c_1 + 2c_2 = 0$ only the last two terms are non zero, thus

$$\lim f(\zeta) = -\frac{1}{5}\left\{\frac{\pi}{2}\;(4c_1s_1 + 4c_2s_2)\right\}.$$

Hence $$\int_0^\infty \frac{\zeta}{\zeta^5+1}\,d\zeta = \left(-\frac{2\pi}{5} + \frac{2\pi}{25}\right)c_1s_1 + \left(-\frac{2\pi}{5} - \frac{6\pi}{25}\right)c_2s_2$$

$$= - \frac{8\pi}{25} (c_1 s_1 + 2c_2 s_2)$$

$\therefore$ $$= \frac{32\pi}{25} \sin^3 \frac{\pi}{5} \cos \frac{\pi}{5}$$

Appendix 2·7: Theoretical aspects of the numerical evaluation of $\lambda_{d\ min}$ and $\lambda_{d\ max}$

·1 Definitions

Since $$S(R) = \frac{\lambda_c |R|}{|R|^n + \lambda_c}$$

it follows that

$$S(k\Delta R) = \Delta R \frac{\lambda_d |k|}{|k|^n + \lambda_d} \quad \text{where} \quad \lambda_d = \lambda_c \Delta R^{-n} .$$

The vector $\underline{S}$ is now defined as

$$(\underline{S})_k = \Delta R \frac{\lambda_d \Lambda_k}{\Lambda_k^n + \lambda_d} \qquad k = 0(1)N_3 - 1$$

(where $\Lambda_k = (\underline{\Lambda})_k$ and $\underline{\Lambda}$ is as defined in Appendix 2·4) and is the discrete counterpart of $S(R)$.

The values of $\lambda_{d\ min}$ and $\lambda_{d\ max}$ are now sought such that

$$\lambda_{d\ min} \leq \lambda_d$$

ensures that the errors due to aliasing in $\underline{s} = \underline{\underline{W}}^{-1}\underline{S}$ are less than a small faction θ_2 of the peak value of $\underline{s}$, and such that

$$\lambda_d \leq \lambda_{d\ max}$$

ensures that the number of samples in each half cycle of $\underline{s}$ is greater than an integer m.

Two vectors $\underline{S}'$ and $\underline{S}''$ are now defined as the discrete version of $S(R)$ sampled with $\Delta R = \Delta R'$ and $\Delta R = \Delta R'' = \Delta R'/32$ whence

$$(\underline{S}')_k = \Delta R' \frac{\lambda_d' \Lambda_k}{\Lambda_k^n + \lambda_d'} \qquad \lambda_d' = \lambda_c \Delta R'^{-n}$$

$$(\underline{S}'')_k = \Delta R'' \frac{\lambda_d'' \Lambda_k}{\Lambda_k^n + \lambda_d''} \qquad \lambda_d'' = \lambda_c \Delta R''^{-n} = \lambda_d' 32^n$$

$$= \frac{\Delta R'}{32} \frac{(32)^n \lambda_d' \Lambda_k}{\Lambda_k^n + (32)^n \lambda_d'}$$

and the corresponding real space vectors $\underline{s}'$ and $\underline{s}''$ are defined by

$$\underline{s}' = \underline{\underline{W}}^{-1}\underline{S}' \quad \text{and} \quad \underline{s}'' = \underline{\underline{W}}^{-1}\underline{S}'' .$$

If, for the original vectors $\underline{s}$ and $\underline{S}$ the number of samples taken was N_3, then this corresponds to sampling $s(r)$ in real space at intervals of $\Delta r = (N_3 \Delta R^{-1})$, thus the sampling intervals used to generate $\underline{s}'$ and $\underline{s}''$ from $s(r)$ are $\Delta r' = (N_3 \Delta R')^{-1}$ and $\Delta r'' = (N_3 \Delta R'')^{-1} = (N_3 \Delta R'/32)^{-1} = 32\Delta r'$. Similarly, $\underline{s}'$ corresponds to sampling $s(r)$ for $|r| \leq (2\Delta R')^{-1}$ while $\underline{s}''$ corresponds to sampling for $|r| \leq (2\Delta R'')^{-1} = 32(2\Delta R')^{-1}$.

N.B. There are minor differences between the definitions of $\underline{s}$ in this section and the definition of $\underline{s}^*$ in §2·541 (and Appendices 2·1 and 2·4). In §2·541, s and S are defined as usual, $\underline{s}^*$ is defined as

$$(\underline{s}^*)_k = \Delta r(s(k\Delta r) + s((k - N_3)\Delta r))$$

so that the components of $\underline{\underline{s}}^* \hat{\underline{p}}$ are estimates at various points of $s * \hat{p}$, and $\underline{S}^*$ is defined as the D.F.T. of $\underline{s}^*$. In this section, s and S are defined as before, but $\underline{S}$ is defined by

$$(\underline{S})_k = S(k\Delta R) + S((k - N_3)\Delta R)$$

and $\underline{s}$ is the D.F.T. of $\underline{S}$. This means that $\underline{s}$ and $\underline{S}$ are constant multiples of $\underline{s}^*$ and $\underline{S}^*$ respectively.

The vectors $\underline{s}$ and $\underline{S}$ are used in preference here as it removes the necessity of carrying an arbitary constant throughout the working and then cancelling it at the end.

·2 Evaluation of $\lambda_{d\,min}$

As seen in §2·5431, the effect of sampling $S(R)$ with ΔR too large is to cause aliasing when estimating $\underline{s}$. The basic method of finding $\lambda_{d\,min}$ is therefore to start with ΔR sufficiently small and gradually increase it until the errors in $\underline{s}$ are too large. Since $\lambda_d = \lambda_c \Delta R^{-n}$, increasing ΔR for any given λ_c is equivalent to decreasing λ_d, and the process of choosing larger and larger ΔR is implemented in practice by choosing smaller and smaller λ_d.

Returning to the vectors $\underline{S}'$ and $\underline{S}''$, by definition $\underline{S}''$ is associated with a sample interval 1/32 nd of that associated with $\underline{S}'$ and it is therefore considered that any aliasing present in $\underline{s}'' = \underline{\underline{W}}^{-1}\underline{S}''$ is negligible when compared with that present in $\underline{s}' = \underline{\underline{W}}^{-1}\underline{S}'$. Accordingly $\underline{s}''$ is considered an accurate estimate of s and $\underline{s}'$. Accordingly $\underline{s}''$ is considered an accurate estimate of s and $\underline{s}'$ an estimate liable to aliasing errors. Further since $\Delta r'' = 32\Delta r'$

$$\Delta R' \sqrt{N_3}\ (\underline{s}')_{32k} \simeq s(32k\Delta r') \qquad\qquad 32k \le N_3/2$$

and $$\Delta R'' \sqrt{N_3}\ (\underline{s}'')_k \simeq s(k\Delta r'') = s(32k\Delta r') \qquad\qquad k \le N_3/2$$

hence $\Delta R' \sqrt{N_3}(\underline{s}')_{32k} \simeq \Delta R'' \sqrt{N_3}(\underline{s}'')_k$

or $$32(\underline{s}')_{32k} \simeq (\underline{s}'')_k \quad .$$

Thus when aliasing is not present in $\underline{s}'$ these two components should be equal, the aliasing error present in $\underline{s}'$ is therefore checked by examining the difference between the corresponding elements of the two vectors.

The strategy for finding $\lambda_{d\ min}$ is therefore to start with λ_d sufficiently large so that the maximum difference between corresponding elements of $\underline{s}'$ and $\underline{s}''$ is less than θ_2 times the largest components of $\underline{s}''$, and then to reduce λ_d until the critical value $\lambda_d = \lambda_{d\ min}$ is reached, recalculating $\underline{s}'$ and $\underline{s}''$ for each new λ_d.

·3 Evaluation of $\lambda_{d\ max}$

As seen in §2·5431, the effect of using λ_c too large, is that the real space sample rate Δr may be insufficient to reproduce $s(r)$ properly. Since $\lambda_d = \lambda_c\ \Delta R^{-n}$ increasing λ_c for fixed ΔR is equivalent to increasing λ_d and the method used to find $\lambda_{d\ max}$ is to start with an acceptable value of λ_d and increase it until insufficient sampling occurs.

By definition, $\underline{s}'$ has a sample interval $\Delta r'$ which is relatively fine compared to the interval $32\Delta r'$ for $\underline{s}''$. Accordingly a graph of $\underline{s}'$ is regarded as a true 'continuous' representation of the graph of s, while that of $\underline{s}''$ is regarded as a discrete plot of an estimate of s which is liable to insufficient sampling. As above

$$32(\underline{s}')_{32k} \simeq (\underline{s}'')_k$$

and graphing $\underline{s}'$ with the corresponding points of $\underline{s}''$ superimposed allows one to see how many samples of $\underline{s}''$ occur in each half cycle of $\underline{s}'$ (i.e. of $s(r)$).

The strategy for finding $\lambda_{d\ max}$ is therefore to start with a value λ_d which gives more than m samples per half cycle of s and then to increase it until this critical value is reached.

·4 Comments

Note that when finding $\lambda_{d\ min}$ it is possible to violate the requirement $\lambda_d < \lambda_{d\ max}$, however this may easily be monitored by observing the graph of $\underline{s}''$ at all times. Similarly when finding $\lambda_{d\ max}$ it is possible to violate $\lambda_d \geq \lambda_{d\ min}$, but provided that the constant $\lambda_{d\ min}$ has been found first, this may also be monitored.

A complete account of the software implementation of this section together with the various results obtained can be found in Appendix 2·8.

Appendix 2·8: Computation aspects of the numerical evaluation of $\lambda_{d\ min}$ and $\lambda_{d\ max}$

·1 Introduction

In §2·54 it was seen that λ must be such that

$$\lambda_{min} \leq \lambda \leq \lambda_{max}$$

if $\underline{s}$ was to be a good representation of s. The reasons for this were as follows. Define S,s,$\underline{S}$,$\underline{s}$ as in §2·7·1.

If $\lambda_{min} < \lambda < \lambda_{max}$ then $\underline{S}$ is representative of S(R) and $\underline{s}$ of s(r), such a case is illustrated in F·1.

If $\lambda > \lambda_{max}$, $\underline{S}$ is a poor representative of S because a significant proportion of the tails of S is truncated, this manifests itself in real space as undersampling of s thus making $\underline{s}$ a poor representative of s (see F·2).

If $\lambda < \lambda_{min}$, then $\underline{S}$ is under sampled and does not properly represent S, or equivalently in real space $\underline{s}$ is an aliased version of s (see F·3).

In order to quantify these problems the ideas developed in §2·54 and §2·7· are implemented as the computer programs:-

TABERR5 PRINTERR5

TABMIN5 ERROR

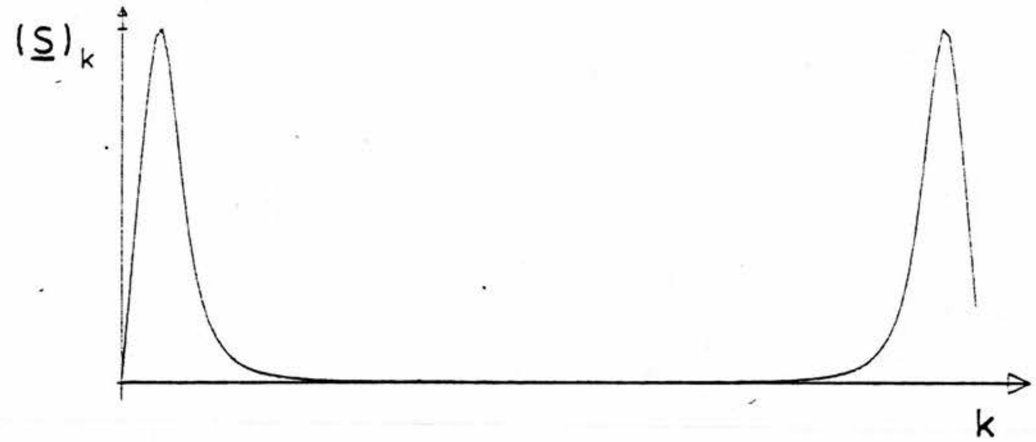
TABMAX5

These are now considered and their outputs presented. (The listings may be found at the end of the appendix.)

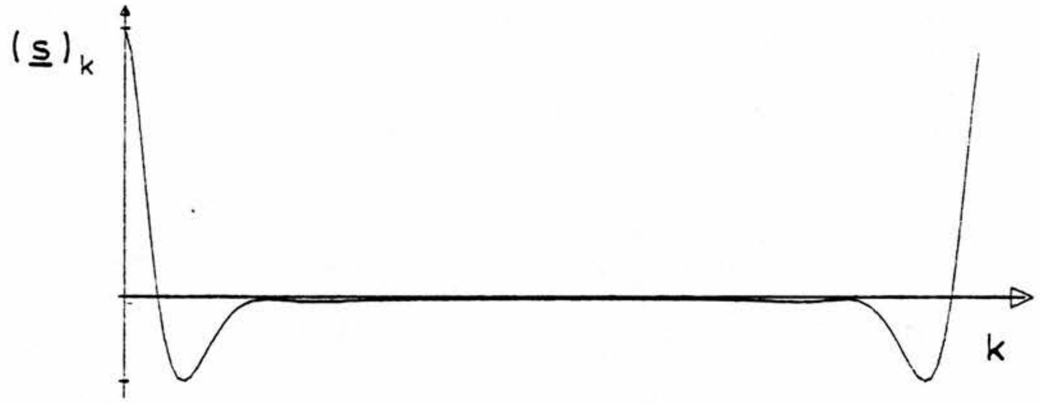
· 2 TABERR5

·21 Method

Consider two different sampled versions of S namely:

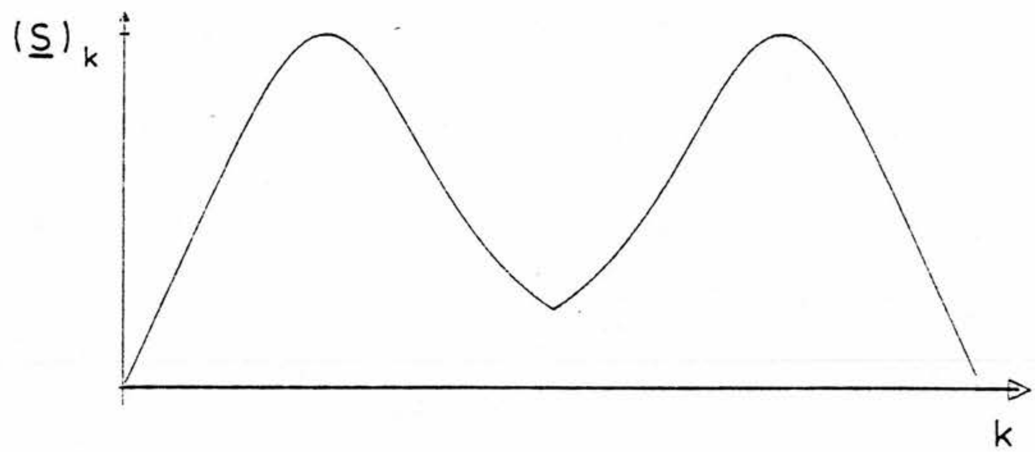


Graph of $\underline{S}$

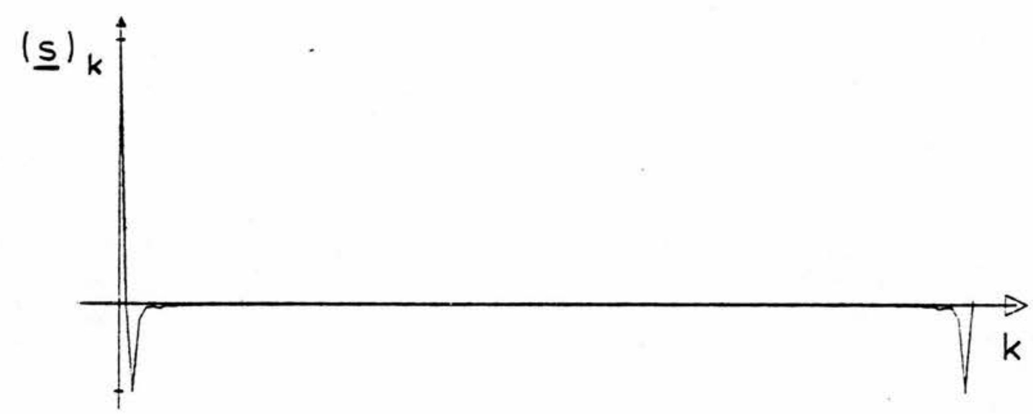


Graph of $\underline{s}$

fig. 2·8·1 : $\lambda_{min} \leqslant \lambda \leqslant \lambda_{max}$

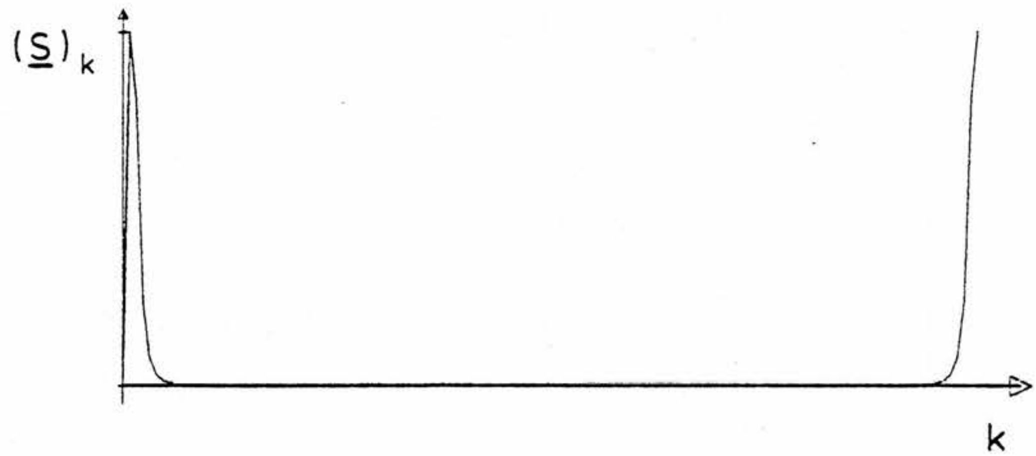


Graph of $\underline{S}$

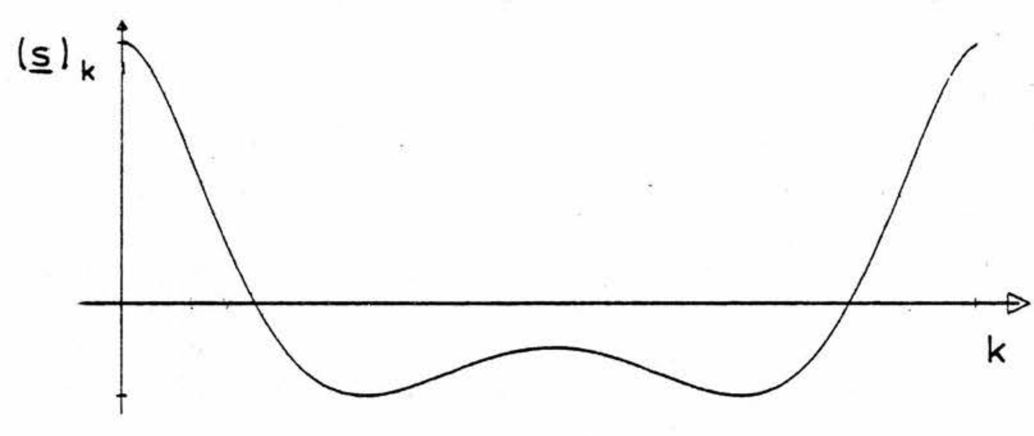


Graph of $\underline{s}$

fig. 2·8·2 : $\lambda > \lambda_{max}$



Graph of $\underline{S}$



Graph of $\underline{s}$

fig. 2·8·3 : $\lambda < \lambda_{min}$

$\underline{S}'$ based on sampling S with $\Delta R = \Delta R'$

and $\underline{S}''$ based on sampling S with $\Delta R = \Delta R'' = \Delta R'/32$

Then (see §2·7·2) any errors in $\underline{S}''$ due to λ being too small are negligible compared to these in $\underline{S}'$. The program ERROR evaluates both $\underline{S}'$ and $\underline{S}''$ and compares their transforms $\underline{s}'$ and $\underline{s}''$ and (assuming $\underline{s}''$ to be error free) outputs the maximum error in $\underline{s}'$ (expressed as a percentage of the maximum element in $\underline{s}''$).

The program TABERR5 is used to make repeated calls on ERROR, and presents the output as a table of error values tabulated for various λ and N (see F·4 and F·5).

·22 Interpretation of output

$\underline{S}'$ and $\underline{S}''$ are both discrete versions of S and thus have different values of λ_d corresponding to the same λ_c, i.e.

$$\lambda_d' = \lambda_c \, \Delta R'^{-n}$$

and $\lambda_d'' = \lambda_c \Delta R''^{-n} = 32^n \lambda_d'$

Now consider what happens as the value of λ_d' is varied. For a medium sized value of λ_d

$$\lambda_{d\ min} \le \lambda_d' < \lambda_d'' \le \lambda_{d\ max}$$

As λ_d' decreases, the inequality $\lambda_d'' \le \lambda_{d\ max}$ remains true but the inequality $\lambda_{d\ min} \le \lambda_d'$ is violated and the error due to aliasing in $\underline{s}'$ increases.

As λ_d' increases, from the same starting value, the inequality $\lambda_{d\ min} \le \lambda_d'$ remains true but $\lambda_d'' \le \lambda_{d\ max}$ is violated, thus $\underline{s}''$ is no longer representative of s as a significant proportion of the tails of S have been truncated when representing it by $\underline{S}''$. Thus once again the error will increase.

This is precisely what is observed in F·4 and F·5. Consider the curve N = 1024 in F·5 as λ_d' increases from 20 to 2000 the error decreases but beyond this $\underline{s}''$ becomes inaccurate and the error increases once more. Thus for

TABULATED VALUES OF ERROR FOR VARIOUS NLAMDAD AND CAPN : LITN=5

NLAMDAD !	CAPN 128	256	512	1024	2048	4096	8192
1	3.836E 1	3.738E 1	3.718E 1	3.719E 1	3.727E 1	3.727E 1	3.727E 1
2	3.722E 1	3.582E 1	3.552E 1	3.547E 1	3.547E 1	3.547E 1	3.547E 1
5	2.785E 1	2.659E 1	2.624E 1	2.619E 1	2.618E 1	2.618E 1	2.618E 1
10	2.293E 1	1.753E 1	1.724E 1	1.719E 1	1.718E 1	1.718E 1	1.718E 1
20	3.990E 1	9.353E 0	9.255E 0	9.217E 0	9.212E 0	9.211E 0	9.211E 0
50	7.783E 1	6.783E 0	3.184E 0	3.073E 0	3.073E 0	3.073E 0	3.072E 0
100	1.215E 2	1.149E 1	2.746E 0	2.663E 0	2.656E 0	2.658E 0	2.658E 0
200	1.827E 2	1.966E 1	2.582E 0	2.360E 0	2.364E 0	2.363E 0	2.364E 0
500	2.996E 2	3.772E 1	3.159E 0	2.069E 0	2.062E 0	2.061E 0	2.061E 0
1000	4.240E 2	6.014E 1	5.704E 0	1.541E 0	1.534E 0	1.533E 0	1.532E 0
2000	5.896E 2	9.374E 1	9.488E 0	1.115E 0	1.064E 0	1.063E 0	1.063E 0
5000	8.940E 2	1.618E 2	1.781E 1	1.654E 0	7.436E -1	7.422E -1	7.420E -1
10000	1.211E 3	2.370E 2	2.839E 1	2.855E 0	5.684E -1	5.669E -1	5.667E -1
20000	1.631E 3	3.390E 2	4.498E 1	4.632E 0	4.912E -1	4.283E -1	4.280E -1
50000	2.397E 3	5.283E 2	8.116E 1	8.498E 0	8.577E -1	2.971E -1	2.969E -1
100000	3.193E 3	7.270E 2	1.239E 2	1.337E 1	1.434E 0	2.253E -1	2.250E -1
200000	4.242E 3	9.898E 2	1.843E 2	2.106E 1	2.286E 0	2.198E -1	1.705E -1
500000	6.152E 3	1.471E 3	2.999E 2	3.849E 1	4.121E 0	4.415E -1	1.182E -1
1000000	8.130E 3	1.972E 3	4.231E 2	6.041E 1	6.400E 0	7.213E -1	9.084E -2

$\lambda_d'' > \lambda_{d\,max}$ $\lambda_d'' \leqslant \lambda_{d\,max}$

fig. 2·8·4

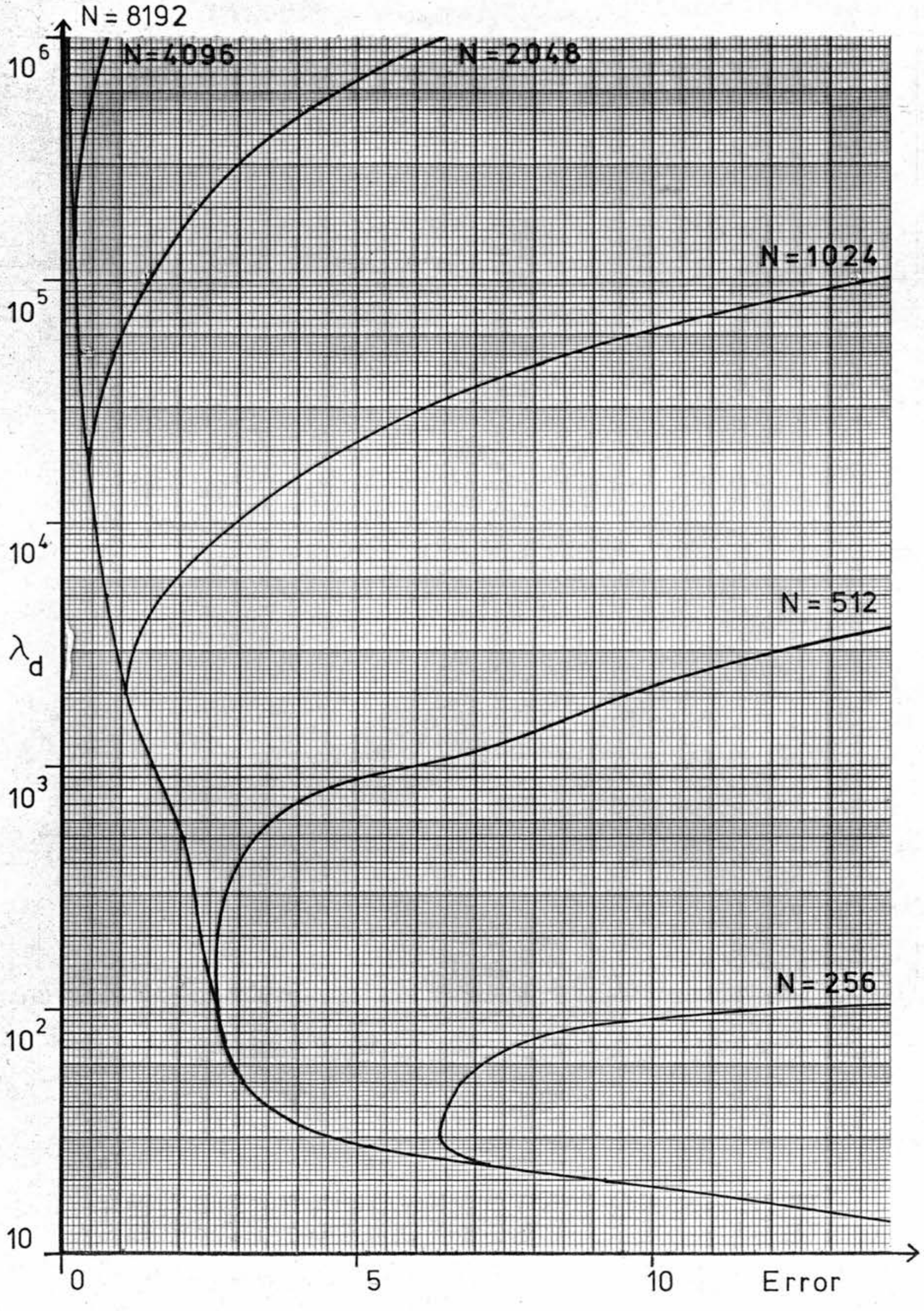


fig. 2·8·5 : Graph of error in $\underline{s}$.

λ_d < 2000 the curve N = 1024 is a graph of $\lambda_{d\ min}$ as a function of error in $\underline{s}'$ but for λ_d > 2000 this is no longer true as the basic assumption that $\underline{s}''$ is accurate has been violated.

It is clear that the graph of $\lambda_{d\ min}$ is just the envelope of the family of curves illustrated in F·5. For practical purposes this is taken to be identical to the curve N = 8192.

·3 TABMIN5

·31 Method

As explained in §·22 the graph of $\lambda_{d\ min}$ against error is identified for practical purposes with the graph of λ_d for N = 8192. This program makes repeated calls on the routine ERROR for N = 8192 and tabulates the results as a table of corresponding values of $\lambda_{d\ min}$ and error and also as a table of results of $\log(\lambda_{d\ min})$ and log(error).

·32 Output

See F2·6 and F2·7.

·4 TABMAX5

·41 Method

This program enables the user to find $\lambda_{d\ max}$ as a function of m and N. The basic idea of the program is as follows.

Consider $\underline{S}$ and $\underline{s}$ for any particular λ_d, as λ_d increases the width of the central peak of $\underline{S}$ (relative to the sample spacing) increases. Correspondingly the width of the central peak of $\underline{s}$ decreases. Suppose it is desired to find the value of $\lambda_{d\ max}$ for m = 3 say, the position of the sample points in $\underline{s}$ on the graph of s is shown in F·6 for $\lambda_d \simeq \lambda_{d\ max}$. It has already been seen in F·1 that

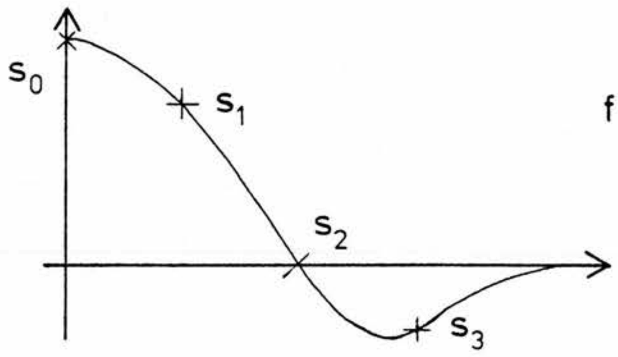


fig. 6a : $\lambda_d = \lambda_{d\ max}$

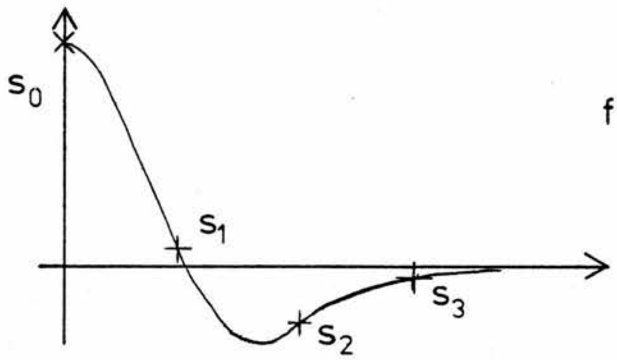


fig. 6b : $\lambda_d > \lambda_{d\ max}$

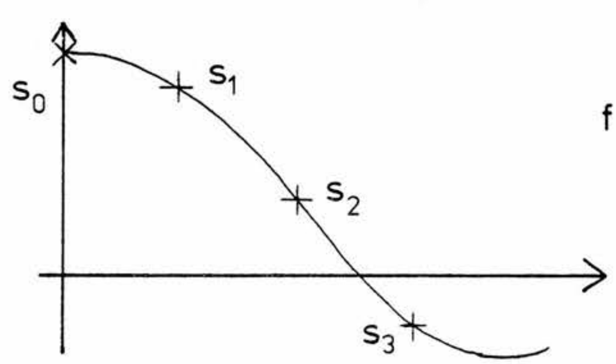


fig. 6c : $\lambda_d < \lambda_{d\ max}$

fig. 2·8·6 : Relation between s(r) and $\underline{s}$.

TABULATED VALUES OF n LAMDA d max/(N/(m-1))**n
n LAMDA d max ,
AND s(m-1) FOR VARIOUS N AND m : LITN= 5

m	CAPN 128	256	512	1024	2048	4096	8192
2	2.873@ -3 9.870@ 7 -0.00094	2.873@ -3 3.159@ 9 0.00063	2.871@ -3 1.010@ 11 0.01636	2.869@ -3 3.230@ 12 0.02699	2.873@ -3 1.035@ 14 0.00447	2.874@ -3 3.313@ 15 -0.00056	2.873@ -3 1.060@ 17 0.00197
3	2.157@ -3 2.316@ 6 -0.00234	2.160@ -3 7.422@ 7 -0.00148	2.156@ -3 2.370@ 9 0.04428	2.161@ -3 7.603@ 10 0.00019	2.161@ -3 2.433@ 12 0.00032	2.161@ -3 7.786@ 13 -0.00051	2.161@ -3 2.491@ 15 0.00334
4	2.185@ -3 3.090@ 5 0.00285	2.193@ -3 9.923@ 6 -0.00005	2.195@ -3 3.178@ 8 0.00012	2.195@ -3 1.017@ 10 0.00319	2.195@ -3 3.255@ 11 0.00082	2.196@ -3 1.042@ 13 -0.00583	2.195@ -3 3.333@ 14 0.00176
5	2.172@ -3 7.288@ 4 -0.00090	2.185@ -3 2.346@ 6 0.00082	2.188@ -3 7.519@ 7 -0.00050	2.189@ -3 2.407@ 9 -0.00066	2.189@ -3 7.703@ 10 -0.00038	2.189@ -3 2.465@ 12 -0.00025	2.189@ -3 7.888@ 13 -0.00015
6	2.164@ -3 2.379@ 4 0.00045	2.184@ -3 7.685@ 5 0.00021	2.189@ -3 2.465@ 7 -0.00007	2.190@ -3 7.892@ 8 0.00131	2.191@ -3 2.526@ 10 -0.00005	2.191@ -3 8.083@ 11 0.00104	2.191@ -3 2.587@ 13 -0.00184
7	2.151@ -3 9.505@ 3 0.00004	2.180@ -3 3.083@ 5 0.00200	2.188@ -3 9.900@ 6 0.00008	2.190@ -3 3.171@ 8 -0.00181	2.191@ -3 1.015@ 10 -0.00299	2.191@ -3 3.248@ 11 -0.00204	2.190@ -3 1.039@ 13 0.00440
8	2.137@ -3 4.369@ 3 0.00024	2.177@ -3 1.424@ 5 0.00510	2.187@ -3 4.579@ 6 0.00039	2.190@ -3 1.467@ 8 -0.00011	2.191@ -3 4.696@ 9 -0.00109	2.191@ -3 1.503@ 11 -0.00315	2.191@ -3 4.809@ 12 -0.00059
9	2.121@ -3 2.224@ 3 -0.00367	2.173@ -3 7.292@ 4 -0.00068	2.186@ -3 2.347@ 6 0.00323	2.189@ -3 7.523@ 7 0.00001	2.190@ -3 2.408@ 9 0.00194	2.190@ -3 7.707@ 10 0.00036	2.190@ -3 2.466@ 12 0.00252
10	2.103@ -3 1.223@ 3 0.00130	2.169@ -3 4.038@ 4 -0.00058	2.185@ -3 1.302@ 6 0.00004	2.189@ -3 4.174@ 7 0.00134	2.190@ -3 1.336@ 9 0.00552	2.191@ -3 4.277@ 10 0.00009	2.191@ -3 1.369@ 12 -0.00408

$\lambda < \lambda_{min}$ 1% | 0·5% $\lambda \geqslant \lambda_{min}$

fig. 2·8·7

s has the form of a single positive peak with long negative tails and $\lambda_{d\ max}$ (for $m = 3$) is defined as the value of λ_d which makes the third sample in $\underline{s}$ coincide with the zero in s (this is illustrated in F·6a). Thus when trying to find $\lambda_{d\ max}$, λ_d is adjusted until this happens. If λ_d is too large then $\underline{s}$ is too narrow and (from the shape of s) the third sample point (s_2) will be negative (see F·6b). If λ_d is too small, $\underline{s}$ is too wide and s_2 will be positive.

Thus to find $\lambda_{d\ max}$ for any given m: $\underline{S}$ is calculated and transformed to give $\underline{s}$ for arbitary λ_d and the sign of s_{m-1} examined, λ_d is reduced if it is negative and increased if it is positive.

By this trial and error approach the program TABMAX5 can be made to zero in on the correct value of $\lambda_{d\ max}$ for any given m,N. Note that this is not done automatically by the program, the program is run interactively and it is up to the user to suggest new values of λ_d. For further details see the program description.

·42 Output

The output from the program tabulates three things as functions of m and N,

$$\lambda_{d\ max} \Big/ \left(\frac{N}{m-1}\right)^5$$

$$\lambda_{d\ max}$$

and s_{m-1}

where s_{m-1} is expressed as a percentage of s_0 (see F·7). It is clear from the tabulated values of

$$\lambda_{d\ max} \left(\frac{N}{m-1}\right)^5$$

that $\lambda_{d\ max}$ can be well approximated by

$$\lambda_{d\ max} \simeq 2{\cdot}190 \times 10^{-3} \left(\frac{N}{m-1}\right)^5$$

(the model suggested by considering the geometry of the functions s and S). The exceptions to this are the top two lines of the table (m = 2 and m = 3) and the bottom left hand corner of the table.

It is intended that in practice only values of λ_d will be used which give use to perhaps 5 or more samples in the first peak of s and there is thus no loss in excluding the possibilities n = 2 or m = 3, these lines of the table are therefore ignored.

In trying to find λ_d such that $s_{m-1} \simeq 0$ when N is small it is possible to violate the constraint $\lambda_d \geq \lambda_{d\ min}$, resulting in S being an under sampled version of S. Consider the value of $\lambda_{d\ min}$: if the maximum allowed error in s is 1% (0.5%) then $\lambda_{d\ min}$ has the value $2{\cdot}321 \times 10^3$ $(1{\cdot}355 \times 10^4)$ (by linear interpolation of the value $\log(\lambda_{d\ min})$ and log (error)) thus the values N = 128, m ≥ 9 (N = 128 m ≥ 7) must be excluded from the table.

The constant in the relationship

$$\lambda_{d\ max} \quad \alpha \left(\frac{N}{m-1}\right)^5$$

is therefore calculated by averaging its tabulated values over the remainder of the table (after excluding m ≤ 3 or N = 128 and m ≥ 7) this gives

$$\lambda_{d\ max} \simeq 2{\cdot}1877 \times 10^{-3} \left(\frac{N}{m-1}\right)^5$$

·5 Implications of $\lambda_{d\ max}$ on calcualtion of $\lambda_{d\ min}$

When calculating $\lambda_{d\ min}$ the error was tabulated for various λ_d and N. For the larger values of λ_d it was noted that $\lambda''_d = 32^n \lambda'_d$ might violate the constraint

$$\lambda''_d \leq \lambda_{d\ max}$$

thus invalidating the basic assumption on which the evaluation of the error was based, namely that s" is an accurate representation of s. The maximum value of λ'_d for which the tabulated values of the error remain correct is therefore determined by

$\lambda'_d \leq \lambda_{d\ max}/32^n$.

Using the model given for $\lambda_{d\ max}$ in §·4 this gives

N	$\lambda_{d\ max} / 32^n$
128	$9{\cdot}22 \times 10^{-3}$
256	$2{\cdot}95 \times 10^{-1}$
512	$9{\cdot}44 \times 10^{0}$
1024	$3{\cdot}02 \times 10^{2}$
2048	$9{\cdot}67 \times 10^{3}$
4096	$3{\cdot}09 \times 10^{5}$
8192	$9{\cdot}90 \times 10^{6}$

for m = 4 (which gives the largest allowed values of λ'_d) and all tabulated points in F·4 for λ_d larger than these values must be excluded. This removes the bottom left corner of the table. These are the values which have already been excluded by the qualitative discussion of the curve shapes in §·22.

·6 Discussion

By considering sources of error in §2·54, it was seen how computer programs might be written to help evaluate λ_{min} and λ_{max}. Using a qualitative understanding of these errors enables one to pick out the graph of λ_{min} against error from the tabulated values of error as a function of λ_d and N. This result then enables one to set up a model for $\lambda_{d\ max}$ in terms of m and N, and this model supports the qualitative reasoning used to evaluate $\lambda_{d\ min}$.

These results for $\lambda_{d\ max}$ and $\lambda_{d\ min}$ were incorporated into the data processing software.

```
%EXTERNALROUTINE TABERR5(%STRING(63) S)
      !
      !
      !
      !
      !      REVISION  00
      !      --------
      !
      !      PURPOSE
      !      -------  THIS ROUTINE IS USED TO MAKE MULTIPLE CALLS TO THE
      !      ROUTINE ERROR FOR LITN=5. THE OUTPUT FROM ERROR IS TABULATED
      !      FOR LISTING ON A LP.
      !
      !      USE
      !      ---  THE ROUTINE IS USED AT COMMAND LEVEL THUS:-
      !
      !                COMMAND:TABERR5(OUT,MAX)
      !
      !      OUT  IS ANY VALID FILENAME OR THE DEVICES .LP OR .TT .
      !      MAX  IS AN OPTIONAL INTEGER, DEFAULT VALUE 8192.
      !
      !      COMMENTS
      !      --------
      !      1)   THE ALIASING ERROR IS CALCULATED FOR NLAMDAD=1,2,5,10,..
      !           1000000 ,  FOR LITN=5 AND FOR CAPN ANY INTEGRAL
      !           POWER OF 2 SUCH THAT  128<=CAPN<=MAX.
      !      2)   FOR FURTHER DETAILS SEE PROGRAM ERROR AND REPORTS.
      !      3)   THE PROGRAM CAUSES TEMPORARY USE OF THE I/O CHANNEL 79.
      !
      !      ORIGINATOR
      !      ----------  S.H.C.HUGHES, DEPT. MEDICAL PHYSICS,R.I.E..
      !
      !
      !
      !
      %CONSTINTEGER LITN=5
      %CONSTINTEGERARRAY NLAMDAD(1:19)=1,2,5,10,20,50,
            100,200,500,1000,2000,5000,10000,20000,50000,
            100000,200000,500000,1000000
      %INTEGER CAPN,I,J,LAST
      %STRING(255) MESSAGE
      %STRING(63) %ARRAY T(1:2)
      !
      !
      MESSAGE="INVALID PARAMETER IN ROUTINE CALL   TABERR5(".S.")"
      !
      !      READ IN PARMS
      !
      DECOMP(S,T,2)
```

```
%IF T(1)="" %THEN T(1)=".LP" %AND ->PARM
%IF CHKFILE(T(1),"")=CHKFILE(T(1),".TT") %AND %C
      CHKFILE(T(1),".LP")=CHKFILE(T(1),".LP==") %THEN CLOSE(MESSAGE)
PARM:%IF T(2)#"" %THEN LAST=NUMBER(T(2)) %ELSE LAST=8192
!
!     SET UP OUTPUT CHANNEL
!
DEFINE("ST79,ZZ#TEMP")
SELECTOUTPUT(79)
SET MARGINS(79,14,132)
!
!     CALCULATE AND OUTPUT NLAMDAD
!
NEWPAGE ; NEWLINES(12)
PRINTSTRING("  TABULATED VALUES OF ERROR FOR VARIOUS NLAMDAD".   %C
     " AND CAPN  :  LITN=5")
NEWLINES(4)
PRINTSTRING("            I") ; SPACES(28) ; PRINTSTRING("CAPN")
NEWLINE
PRINTSTRING("    NLAMDAD I")
CAPN=128
%CYCLE
     %EXIT %IF CAPN>LAST
     WRITE(CAPN,10)
     SPACES(2)
     CAPN=CAPN<<1
     %REPEAT
NEWLINE
%CYCLE I=1,1,105
     PRINTSYMBOL('=')
     %REPEAT
NEWLINES(2)
J=0
%CYCLE I=1,1,19
     PRINT(NLAMDAD(I),10,0)
     SPACES(3)
     CAPN=128
     %CYCLE
          %EXIT %IF CAPN>LAST
          SPACES(3)
          PRINTFL(ERROR(NLAMDAD(I),LITN,CAPN,""),3)
          CAPN=CAPN<<1
          %REPEAT
     NEWLINE
     NEWLINE %AND J=J+1 %IF MOD(3*INT((I)/3)-(I))<0.01
     %REPEAT
NEWPAGE
!
!     TIDY UP OUTPUT CHANNEL
```

```
!
SELECTOUTPUT(0)
CLOSESTREAM(79)
CLEAR("ST79")
%IF T(1)->(".").T(2) %THEN LIST("ZZ#TEMP,".T(1)) %C
      %ELSE COPYFILE("ZZ#TEMP,".T(1))
DESTROY("ZZ#TEMP")
!
!
!
%END
```

```
%EXTERNALROUTINE TABMAX5(%STRING(63) S)
      !
      !
      !
      !      REVISION  00
      !      --------
      !
      !      PURPOSE
      !      -------  THIS ROUTINE IS USED FOR TABULATING VALUES OF  :-
      !
      !               N LAMDA D MIN /(N/(M-1))**N  FOR  N = 1 OR 5
      !                                                 N = 128 - 8192
      !                                            AND  M = 2 - 10
      !
      !      USE
      !      ---
      !                    COMMAND;TABMAX5(OUT1,OUT2,GRAPHFILE,CAPSW)
      !
      !      OUT1     IS ANY VALID FILENAME OR THE DEVICES .LP, OR .TT AND IS
      !               USED TO GIVE A LISTING OF THE RESULTS OBTAINED BY THE
      !               PROGRAM.  DEFAULT VALUE .LP.
      !      OUT2     IS ANY VALID FILENAME AND IS USED TO STORE INTERMEDIATE
      !               RESULTS ARISING FROM THE PROGRAM.  IF THE FILE DOES
      !               NOT EXIST ON ENTRY TO THE ROUTINE IT IS CREATED AND
      !               INITIALISED, IF IT ALREADY EXISTS IT IS ASSUMED TO HAVE
      !               BEEN CREATED BY THE PROGRAM AND INTERPRETED ACCORDINGLY.
      !      GRAPHFILE IF GRAPHFILE#"" AND THE LAST VALUE OF N IS LESS
      !               THAN OR EQUAL TO 8192 THEN A GRAPH OF S AND S IS CREATED
      !               IN THE GIVEN FILE.
      !      CAPSW    IF CAPSW="NULL" THEN THE CAPTIONS ON THE GRAPH IS
      !               OMITTED.
      !
      !      COMMENTS
      !      --------
      !      1)    THE PROGRAM MAKES TEMPORARY USE OF I/O CHANNELS 79 AND
      !            80 AND FILES ZZ#TEMP, ZZ#GRAPH AND ZZ#ERMAX.
      !
      !
      !      ORIGINATOR
      !      ----------  S.H.C. HUGHES, DEPT. MEDICAL PHYSICS, R.I.E..
      !
      !
      !
      !
      %CONSTINTEGER LITN=5
      %INTEGER CAPN,I,J,SWITCH,M1,ICONST,X,LITM,CAPSW
      %LONGREAL NLAMDAD,TEMP
      %STRING(8) FILE
      %STRING(255) MESSAGE
```

```
%STRING(63) %ARRAY T(1:4)
%RECORDFORMAT RF(%STRING(8) U,%BYTEINTEGER I,%%STRING(8) S,  %C
                %STRING(15) L,M,%STRING(19) DELIVER)
%RECORDNAME INFO(RF)
%LONGREALARRAY FILE1,FILE2,XDATA(0:8191)
%LONGREALARRAYNAME RATIO,LAMDAMIN,LITS
%LONGREALARRAYFORMAT RFORM(2:10,7:13)
%EXTERNALINTEGERFNSPEC SSINFO
%LONGREALFNSPEC ZERO POINT
!
MESSAGE="INVALID PARAMETER IN ROUTINE CALL   TABMAX5(".S.")"
!
!     READ IN PARMS
!
DECOMP(S,T,4)
%IF T(1)="" %THEN T(1)=".LP" %ELSE %START
      %IF CHKFILE(T(1),"")=CHKFILE(T(1),".TT") %AND %C
            CHKFILE(T(1),".LP")=CHKFILE(T(1),".LP==")  %C
            %THEN CLOSE(MESSAGE)
      %FINISH
%IF (T(2)#"" %AND CHKFILE(T(2),"")="BAD") %OR T(2)="ZZ#TEMP"   %C
      %THEN CLOSE(MESSAGE)
%IF T(3)#"" %AND CHKFILE(T(3),"")=CHKFILE(T(3),".GP")   %C
      %THEN CLOSE(MESSAGE)
%IF T(2)="" %THEN %START
      FILE="ZZ#ERMAX"
      DESTROY(FILE) %IF EXIST(FILE)#0
      SWITCH=0
      %FINISH %ELSE %START
      FILE=T(2)
      SWITCH=EXIST(FILE)
      %FINISH
%IF T(4)="NULL" %THEN CAPSW=0 %ELSE CAPSW=1
RATIO==ARRAY(ADSMFILE(FILE,80,9*7*8*3),RFORM)
LAMDAMIN==ARRAY(ADDR(RATIO(10,13))+8,RFORM)
LITS==ARRAY(ADDR(LAMDAMIN(10,13))+8,RFORM)
%IF SWITCH=0 %START
      %CYCLE M1=7,1,13
            %CYCLE LITM=2,1,10
                  RATIO(LITM,M1)=-1
                  %REPEAT
            %REPEAT
      %FINISH
!
!     READ LITN, LITM AND CAPN
!
CAPNLITM:
PROMPT("LITM? : ") ; READ(LITM) %UNTIL LITM>1
PROMPT("N? : ") ; READ(CAPN) %UNTIL CAPN>0 %AND BITS(CAPN)=1
```

```
ICONST=CAPN ; M1=0
ICONST=ICONST>>1 %AND M1=M1+1 %UNTIL ICONST=0
!
!      READ NLAMDAD
!
LAMDA:
NEWLINE
PROMPT("N LAMDA D? : ") ; READ(NLAMDAD) %UNTIL NLAMDAD>0
!
!      CALCULATE ZERO POINT
!
TEMP=ZERO POINT
PRINTSTRING("S(M-1) = ") ; PRINT(TEMP,3,5) ; NEWLINE
!
!      STORE?
!
%IF 10>=LITM>=2 %AND 8<=M1<=14 %START
     PROMPT("STORE? : ") ; READSYMBOL(X) %UNTIL X='Y' %OR X='N'
     %IF X='N' %THEN ->LAMDA
     NEWLINE
     PRINTSTRING("M = ") ; WRITE(LITM,1)
     PRINTSTRING("      N = ") ; WRITE(CAPN,1) ; NEWLINE
     RATIO(LITM,M1-1)=NLAMDAD/(CAPN/(LITM-1))**LITN
     LAMDAMIN(LITM,M1-1)=NLAMDAD
     LITS(LITM,M1-1)=TEMP
     %FINISH
!
!      GIVE TABULATED OUTPUT
!
PROMPT("EXIT? : ") ; READSYMBOL(X) %UNTIL X='N' %OR X='Y'
->CAPNLITM %IF X='N'
DEFINE("ST79,ZZ#TEMP")
SELECTOUTPUT(79)
SET MARGINS(79,18,132)
NEWPAGE ; NEWLINES(8)
PRINTSTRING("  TABULATED VALUES OF  N LAMDA D MAX/(N/(M-1))**N ")
NEWLINE
PRINTSTRING("                              N LAMDA D MAX ,")
NEWLINE
PRINTSTRING("                         AND  S(M-1)      ".  %C
     " FOR VARIOUS N AND M  ;  LITN=")
WRITE(LITN,1)
NEWLINES(4)
PRINTSTRING("    !") ; SPACES(28) ; PRINTSTRING("CAPN")
NEWLINE
PRINTSTRING("  M !")
%CYCLE M1=7,1,13
     WRITE(1<<M1,10)
     SPACES(2)
```

```
          %REPEAT
    NEWLINE
    %CYCLE I=1,1,93
          PRINTSYMBOL('=')
          %REPEAT
    NEWLINES(2)
    J=0
    %CYCLE LITM=2,1,10
          SPACES(6)
          %CYCLE M1=7,1,13
                SPACES(3)
                %IF RATIO(LITM,M1)>=0 %THEN PRINTFL(RATIO(LITM,M1),3) %C
                      %ELSE SPACES(10)
                %REPEAT
          NEWLINE
          PRINT(LITM,2,0)
          SPACES(3)
          %CYCLE M1=7,1,13
                SPACES(3)
                %IF RATIO(LITM,M1)>=0 %THEN PRINTFL(LAMDAMIN(LITM,M1),3) %C
                      %ELSE PRINTSTRING("    **    ")
                %REPEAT
          NEWLINE
          SPACES(6)
          %CYCLE M1=7,1,13
                SPACES(3)
                %IF RATIO(LITM,M1)>=0 %THEN PRINT(LITS(LITM,M1),3,5)  %C
                      %ELSE SPACES(10)
                %REPEAT
          NEWLINES(2)
          %REPEAT
    NEWPAGE
    !
    !     TIDY UP OUTPUT CHANNEL
    !
    SELECTOUTPUT(0)
    CLOSESTREAM(79)
    CLEAR("ST79")
    %IF T(1)->(".").T(1) %THEN LIST("ZZ#TEMP,.".T(1)) %C
          %ELSE COPYFILE("ZZ#TEMP,".T(1))
    DESTROY("ZZ#TEMP")
    CLOSESM(80)
    CLEAR("SM80")
    DESTROY(FILE) %IF T(2)=""
    !
    !     SET UP GRAPHPLOTTER FILE IF REQUESTED
    !
    %IF T(3)#"" %START
          DEFINE("SQ80,ZZ#GRAPH,,F80")
```

```
      OPENPLOTTER(80)
      INFO==RECORD(SSINFO)
      PLOTSTRING(INFO_DELIVER)
!
!      PLOT OUTPUT
!
      %CYCLE I=0,1,CAPN-1
            XDATA(I)=I ; %REPEAT
      SETPLOT(1.5,1.5,21,30,0)
      DRAW(4,21,13,5.5,0,XDATA,FILE1,0,CAPN-1,"L","O", %C
            "GRAPH OF  S",CAPSW,6,"")
      DRAW(4,11,13,5.5,0,XDATA,FILE2,0,CAPN-1,"L","O", %C
            "GRAPH OF  S",CAPSW,6,"")
      %IF CAPSW#0 %START
            SCALE(6,6.5,0.15,1,0)
            ANNOTATE(0,0,1,0)
            PLOTSTRING("GRAPH OF  S  AND  S  ;  N3  = ")
            PLOTNUMBER(CAPN,1,0)
            PLOTSYMBOL(NL)
            PLOTSTRING("                          ")
            PLOTSYMBOL(X'F') ; PLOTSYMBOL('N')
            PLOTSYMBOL(X'E') ; PLOTSYMBOL(X'AB')
            PLOTSYMBOL(X'F') ; PLOTSYMBOL('D')
            PLOTSYMBOL(X'E') ; PLOTSTRING("  = ")
            PLOTNUMBER(NLAMDAD,0,3)
            PLOTSYMBOL(NL)
            PLOTSTRING("                          ")
            PLOTSTRING("N   = ") ; PLOTNUMBER(LITN,1,0)
            %FINISH
      CLOSEPLOTTER
      CLEAR("SQ80")
      %IF T(3)=".GP" %THEN SEND("ZZ#GRAPH,.GP") %ELSE %START
            COPYFILE("ZZ#GRAPH,".T(3))
            DESTROY("ZZ#GRAPH")
            %FINISH
      %FINISH
!
!
!
%LONGREALFN ZERO POINT
      %LONGREALARRAY CAPS,ZERO(0:CAPN-1),WORK(1:M1)
      %CYCLE I=0,1,CAPN-1
            CAPS(I)=NLAMDAD*CAPLAMDA(I,CAPN)/  %C
                  (CAPLAMDA(I,CAPN)**LITN +NLAMDAD)
            %REPEAT
      %IF T(3)#"" %AND CAPN<=8192 %START
            %CYCLE I=0,1,CAPN-1
                  FILE1(I)=CAPS(I)
                  %REPEAT
```

```
          %FINISH
     RLMATO(ZERO,0,CAPN-1)
     C06ABF(CAPS(0),ZERO(0),CAPN,REVERSE,M1,WORK(1))
     %IF T(3)#"" %AND CAPN<=8192 %START
          %CYCLE I=0,1,CAPN-1
               FILE2(I)=CAPS(I)
               %REPEAT
          %FINISH
     %RESULT=100*CAPS(LITM-1)/CAPS(0)
     %END

!
!
!
%END
```

```
%EXTERNALROUTINE TABMIN5(%STRING(63) S)
     !
     !
     !
     !
     !      REVISION  00
     !      --------
     !
     !      PURPOSE
     !      -------  THIS ROUTINE TABULATES VALUES OF LAMDAMIN FLOR LITN=5
     !      IT ASSUMES THAT THE GRAPH OF LAMDAMIN IS IDENTICAL TO THAT OF
     !      LAMDA FOR CAPN=8192.
     !
     !      USE
     !      ---  THE ROUTINE IS USED AT COMMAND LEVEL THUS:-
     !
     !                COMMAND:TABMIN5(OUT)
     !
     !      OUT  IS ANY VALID FILENAME OR THE DEVICES .LP OR .TT .
     !
     !      COMMENTS
     !      --------
     !      1)   THE PROGRAM CAUSES TEMPORARY USE OF THE I/O CHANNEL 79.
     !
     !      ORIGINATOR
     !      ----------  S.H.C.HUGHES, DEPT. MEDICAL PHYSICS,R.I.E..
     !
     !
     !
     !
     %CONSTLONGREALARRAY LAMDA(1:22)=0.1,0.2,0.5,1,2,5,10,20,50,100,200,500,
                               1000,2000,5000,10000,20000,50000,
                               100000,200000,500000,1000000
     %INTEGER I
     %LONGREAL ERR
     %STRING(255) MESSAGE
     !
     !
     MESSAGE="INVALID PARAMETER IN ROUTINE CALL   LAMMIN5(".S.")"
     !
     !
     !      READ IN PARMS
     !
     %IF S="" %THEN S=".LP"
     %IF CHKFILE(S,"")=CHKFILE(S,".TT") %AND %C
          CHKFILE(S,".LP")=CHKFILE(S,".LP==") %THEN CLOSE(MESSAGE)
     !
     !      SET UP OUTPUT CHANNEL
     !
```

```
DEFINE("ST79,ZZ#TEMP")
SELECTOUTPUT(79)
SET MARGINS(79,25,132)
!
!     CALCULATE AND OUTPUT NLAMDAD
!
NEWPAGE ; NEWLINES(8)
PRINTSTRING("TABULATED VALUES FOR LAMDA MIN AND ERROR   :   LITN=5")
NEWLINES(3)
PRINTSTRING("LAMDA MIN           ")
PRINTSTRING("ERR                 ")
PRINTSTRING("LOG ( LAMDA MIN )   ")
PRINTSTRING("LOG ( ERR )         ")
NEWLINE
%CYCLE I=1,1,80
     PRINTSYMBOL('=')
     %REPEAT
NEWLINES(3)
%CYCLE I=1,1,22
     ERR=ERROR(LAMDA(I),5,8192,"")
     PRINTFL(LAMDA(I),10)       ; SPACES(3)
     PRINTFL(ERR,10)            ; SPACES(3)
     PRINTFL(LOG(LAMDA(I)),10)  ; SPACES(3)
     PRINTFL(LOG(ERR),10)       ; SPACES(3)
     NEWLINE
     NEWLINE %IF MOD(3*INT((I)/3)-(I))<0.01
     %REPEAT
NEWPAGE
!
!     TIDY UP OUTPUT CHANNEL
!
SELECTOUTPUT(0)
CLOSESTREAM(79)
CLEAR("ST79")
%IF S->(".").S %THEN LIST("ZZ#TEMP,,".S) %C
     %ELSE COPYFILE("ZZ#TEMP,".S)
DESTROY("ZZ#TEMP")
!
!
!
%END
```

```
%EXTERNALROUTINE PRINTERR(%STRING(63) S)
     !
     !
     !      REVISION  00
     !      --------
     !
     !      PURPOSE
     !      -------
     !      THIS ROUTINE IS USED TO CALL THE LONGREALFN ERROR AT COMMAND
     !      LEVEL.
     !
     !      USE
     !      ---  THE ROUTINE IS WRITTEN AS AN EXTERNALROUTINE AND  USED
     !      AT COMMAND LEVEL THUS:-
     !
     !           COMMAND:PRINTERR(LITN,CAPN,GRAPHFILE)
     !
     !      LITN      HAS THE VALUE  1 OR 5 DEPENDING ON THE FILTER FN
     !                CONSIDERED.
     !      CAPN      IS AN INTEGRAL POWER OF TWO AND IS THE NUMBER
     !                OF DATA POINTS USED IN THE FFT.WHEN CALCULATING LITS
     !                FROM CAPS.
     !      GRAPHFILE IS AN OPTIONAL PARAMETER,IF GIVEN IT IS USED AS
     !                A FILE FOR A GRAPHPLOTTER OUTPUT, THE FILE CONTAINS
     !                GRAPHS OF THE ALISED FUNCTION LITSD AND THE ERROR FUNCTION
     !                (IN %) ASSOCIATED WITH IT.
     !
     !      WHEN RUN, THE PROGRAM DEMANDS A VALUE FOR NLAMDAD,AND OUTPUTS
     !      THE MAXIMUM ERROR THETA IN LITSD.
     !
     !      COMMENTS
     !      --------
     !      1)   FOR FURTHER DETAILS SEE REPORT.
     !      2)   IF GRAPHFILE#"" THEN I/O CHANNEL 80 IS USED.
     !
     !      ORIGINATOR
     !      ----------  S.H.C. HUGHES, DEPT. MEDICAL PHYSICS, R.I.E.
     !
     !
     !
     !
     %INTEGER LITN,CAPN
     %LONGREAL NLAMDAD
     %STRING(63) %ARRAY T(1:3)
     !
     !
     !
     DECOMP(S,T,3)
     LITN=NUMBER(T(1))
```

```
CAPN=NUMBER(T(2))
PROMPT("NLAMDAD=")
READ(NLAMDAD)
PRINTSTRING("ERROR  =")
PRINTFL(ERROR(NLAMDAD,LITN,CAPN,T(3)),3)
%END
```

```
%EXTERNALLONGREALFN ERROR(%LONGREAL NLAMDAD,%INTEGER LITN,CAPN,  %C
                          %STRING(24) GRAPHFILE)
    !
    !
    !
    !     REVISION 00
    !     --------
    !
    !     PURPOSE
    !     -------  THIS ROUTINE WILL EVALUATE THE ALIASING ERROR THETA
    !     CAUSED BY INSUFFICIENT SAMPLING OF THE FUNCTION S.
    !
    !     USE
    !     ---  THE ROUTINE IS WRITTEN AS AN EXTERNALROUTINE AND  USED
    !     AT COMMAND LEVEL THUS;-
    !
    !          COMMAND:ERROR(NLAMDAD,LITN,CAPN,GRAPHFILE)
    !
    !     NLAMDAD   IS THE VALUE OF THE PARAMETER LAMDA IN THE
    !               FN S.
    !     LITN      HAS THE VALUE  1 OR 5 DEPENDING ON THE FILTER FN
    !               CONSIDERED.
    !     CAPN      IS AN INTEGRAL POWER OF TWO AND IS THE NUMBER
    !               OF DATA POINTS USED IN THE FFT.WHEN CALCULATING LITS
    !               FROM CAPS.
    !     GRAPHFILE IS AN OPTIONAL PARAMETER.IF GIVEN IT IS USED AS
    !               A FILE FOR A GRAPHPLOTTER OUTPUT. THE FILE CONTAINS
    !               GRAPHS OF THE ALISED FUNCTION LITSD AND THE ERROR FUNCTION
    !               (IN %) ASSOCIATED WITH IT.
    !
    !     COMMENTS
    !     --------
    !     1)   FOR FURTHER DETAILS SEE REPORT.
    !     2)   IF GRAPHFILE#"" THEN I/O CHANNEL 80 IS USED.
    !
    !     ORIGINATOR
    !     ----------  S.H.C. HUGHES, DEPT. MEDICAL PHYSICS, R.I.E.
    !
    !
    !
    !
    %CONSTINTEGER REVERSE=0,K=5
    %LONGREAL RESULT
    %INTEGER ICONST,M1,I
    %STRING(255) MESSAGE
    %RECORDFORMAT RF(%STRING(8) U,%BYTEINTEGER I,%%STRING(8) S,  %C
                     %STRING(15) L,M,%STRING(19) DELIVER)
    %RECORDNAME INFO(RF)
    %LONGREAL MAX,MIN,MAXSDD
```

```
%EXTERNALINTEGERFNSPEC SSINFO
!
!
!
MESSAGE="INVALID PARAMETER IN ROUTINE CALL     ERROR"
!
!
!     READ IN PARMS
!
%IF LITN#1 %AND LITN#5 %THEN CLOSE(MESSAGE)
%IF BITS(CAPN)#1 %OR CAPN>>(K+2)=0 %THEN CLOSE(MESSAGE)
%IF GRAPHFILE#"" %AND CHKFILE(GRAPHFILE,"")=CHKFILE(GRAPHFILE,".GP") %C
     %THEN CLOSE(MESSAGE)
ICONST=CAPN ; M1=0
ICONST=ICONST>>1 %AND M1=M1+1 %UNTIL ICONST=0
!
!     SET UP DATA ARRAYS
!
%BEGIN
     %LONGREALARRAY WORK(1:M1),ERROR,ALIASED,TRUE,XDATA(0:CAPN>>K-1)
     %LONGREALFNSPEC THETA(%LONGREAL NLAMDAD)
     %ROUTINESPEC SELECT(%LONGREALARRAYNAME ARRAY1,ARRAY2,%INTEGER I)
     !
     !     FIND THETA
     !
     RESULT=THETA(NLAMDAD)
     !
     !     SET UP GRAPHPLOTTER FILE IF REQUESTED
     !
     %IF GRAPHFILE#"" %START
          DEFINE("SQ80,ZZ#GRAPH,,F80")
          OPENPLOTTER(80)
          INFO==RECORD(SSINFO)
          PLOTSTRING(INFO_DELIVER)
     !
     !     PLOT OUTPUT
     !
          %CYCLE I=0,1,CAPN>>K-1
               XDATA(I)=I ; %REPEAT
          SETPLOT(0,0,21,30,0)
          DRAW(4,21,13,5.5,0,XDATA,ALIASED,0,CAPN>>K-1,"L","O", %C
               "  --  S'                    XX  S''  ",1,6,"")
          DRAW(4,21,13,5.5,0,XDATA,TRUE,0,CAPN>>K-1,"P","O", %C
               "",0,6,"O")
          DRAW(4,11,13,5.5,0,XDATA,ERROR,0,CAPN>>K-1,"L","O", %C
               "GRAPH OF % ERRORS IN S'",1,6,"")
          SCALE(6,6.5,0.15,1,0)
          ANNOTATE(0,0,1,0)
          PLOTSTRING("N    =") ; PLOTNUMBER(LITN,4,0)
```

```
        PLOTSTRING("     N LAMDA D     =") ; PLOTNUMBER(NLAMDAD,0,3)
        PLOTSYMBOL(NL)
        PLOTSTRING("N3   =") ; PLOTNUMBER(CAPN,4,0)
        PLOTSTRING("     MAXIMUM ERROR =")
        PLOTNUMBER(MAX,0,3)
        PLOTSYMBOL(NL)
        PLOTSTRING("NLAMDAD**(1/N)/N3   =   ")
        PLOTNUMBER(EXP((1/LITN)*LOG(NLAMDAD))/CAPN,0,3)
        CLOSEPLOTTER
        CLEAR("SQ80")
        %IF GRAPHFILE=".GP" %THEN SEND("ZZ#GRAPH,.GP") %ELSE %START
           COPYFILE("ZZ#GRAPH,".GRAPHFILE)
           DESTROY("ZZ#GRAPH")
           %FINISH
        %FINISH
    !
    !
    !
    !
    !
    %LONGREALFN THETA(%LONGREAL NLAMDAD)
       !
       !    SET UP LITSD
       !
       %BEGIN
          %LONGREALARRAY CAPSD,ZERO(0:CAPN-1)
          %CYCLE I=0,1,CAPN-1
             CAPSD(I)=(NLAMDAD*CAPLAMDA(I,CAPN))/ %C
                      (CAPLAMDA(I,CAPN)**LITN+NLAMDAD)
             %REPEAT
          RLMATO(ZERO,0,CAPN-1)
          C06ABF(CAPSD(0),ZERO(0),CAPN,REVERSE,M1,WORK(1))
          %IF LITN=1 %THEN CAPSD(0)=CAPSD(0)-NLAMDAD*SQRT(CAPN)
          SELECT(CAPSD,ALIASED,CAPN>>K-1)
          %END

       !
       !    SET UP LITSDD
       !
       %BEGIN
          %LONGREALARRAY CAPSDD,ZERO(0:CAPN-1)
          %CYCLE I=0,1,CAPN-1
             CAPSDD(I)=(2**(K*(LITN-1))*NLAMDAD*CAPLAMDA(I,CAPN))/ %C
                      (CAPLAMDA(I,CAPN)**LITN+2**(K*LITN)*NLAMDAD)
             %REPEAT
          RLMATO(ZERO,0,CAPN-1)
          C06ABF(CAPSDD(0),ZERO(0),CAPN,REVERSE,M1,WORK(1))
          %IF LITN=1 %THEN CAPSDD(0)=CAPSDD(0)-NLAMDAD*SQRT(CAPN)
          TRUE(0)=CAPSDD(0)/2**K
```

```
                TRUE(CAPN>>(K+1))=CAPSDD(CAPN>>(K+1))/2**K
                %CYCLE I=1,1,CAPN>>(K+1)-1
                     TRUE(I)=CAPSDD(I)/2**K
                     TRUE(CAPN>>K-I)=CAPSDD(CAPN-I)/2**K
                     %REPEAT
                MAXSDD=RLMAX(TRUE,0,CAPN>>K-1)
                %END

          !
          !     CALCULATE ERROR VALUES
          !
          %CYCLE I=0,1,CAPN>>K-1
               ERROR(I)=(ALIASED(I)-TRUE(I))*100/MAXSDD
               %REPEAT
          !
          !
          !
          MAX=RLMAX(ERROR,0,CAPN>>K-1)
          MIN=-RLMIN(ERROR,0,CAPN>>K-1)
          MAX=MIN %IF MAX<MIN
          %RESULT=MAX
          %END

     !
     !
     !
     !
     !
     %ROUTINE SELECT(%LONGREALARRAYNAME ARRAY1,ARRAY2,%INTEGER N)
          !
          !
          !     PURPOSE
          !     -------  THIS ROUTINE INSERTS EVERY 2**KTH ELEMENT OF ARRAY1
          !     INTO ARRAY2.
          !
          !
          %CYCLE I=0,1,N
               ARRAY2(I)=ARRAY1(I<<K)
               %REPEAT
          %END

     %END

%RESULT=RESULT
%END
```

Appendix 3·1: ZALIAS and its output

This appendix contains:-

1) A listing of the software ZALIAS used to estimate aliasing errors as described in §3·2.
2) A listing of the output from the program.

The subroutine S17ABF is a function subroutine which returns the value $J_1(x)$.

```
FILE IDENTIFIER : IALIAS_ZALSPROG    LISTED ON 02/12/77 AT 14.53.10


%BEGIN
!
!
!
!      THIS PROGRAM WILL FIND THE VALUE OF P FOR WHICH
!
!      P=MAX(!2*(J1(X)/X)*(SIN(X/K)/(X/K))!)
!
!      WHERE  X>=K*PI/2
!      OUTPUT IS VIA ST80. THE BESSEL FN IS EVALUATED USING THE
!      NAG ROUTINE S17ABF.
!
!      ORIGINATOR   S.H.C.HUGHES
!
!      REVISION  00
!
!
!
%INTEGER P,I,J
%REAL K,KMIN,KMAX,INC,INCK,BEGIN,END,Y,MAX,UNIT
%LONGREAL X
%EXTERNALLONGREALFNSPEC S17ABF(%LONGREALNAME X)
!
!
!
SELECTOUTPUT(80)
SETMARGINS(80,30,120)
NEWLINES(2)
SPACES(4);PRINTSTRING('K')
SPACES(9);PRINTSTRING('P-1')
SPACES(7);PRINTSTRING('P1')
SPACES(8);PRINTSTRING('P-2')
NEWLINE
!
!
!
INCK=0.25000000000
INC =0.010000000
KMIN=2.50000000
KMAX=16.0000000
!
!
I=0
K=KMIN
%WHILE K<=KMAX %CYCLE
NEWLINE
PRINT(K,4,4)
UNIT=K*PI/2
%CYCLE P=1,1,3
BEGIN=P*UNIT
END=BEGIN+UNIT
MAX=0.00000000
J=0
X=BEGIN
%WHILE X<=END %CYCLE
      Y=!2*SIN(X/K)*S17ABF(X)*K/(X*X)!
      %IF Y>MAX %THEN MAX=Y
      J=J+1
      X=BEGIN+J*INC
      %REPEAT
PRINT(MAX,4,4)
%REPEAT
I=I+1
K=KMIN+I*INCK
%REPEAT
PRINTSTRING('REQUIRED VALUES FOUND')
NEWLINES(2)
SELECTOUTPUT(0)
CLOSESTREAM(80)
%ENDOFPROGRAM
```

FILE IDENTIFIER : ZALIAS_ZALEDATA LISTED ON 02/12/77 AT 14.55.06

K	P-1	P1	P-2
2.5000	0.0609	0.0086	0.0084
2.7500	0.0705	0.0084	0.0043
3.0000	0.0785	0.0071	0.0056
3.2500	0.0842	0.0061	0.0052
3.5000	0.0791	0.0059	0.0038
3.7500	0.0641	0.0052	0.0044
4.0000	0.0430	0.0045	0.0029
4.2500	0.0305	0.0044	0.0032
4.5000	0.0335	0.0040	0.0035
4.7500	0.0361	0.0036	0.0024
5.0000	0.0384	0.0034	0.0027
5.2500	0.0405	0.0032	0.0025
5.5000	0.0401	0.0029	0.0022
5.7500	0.0340	0.0028	0.0023
6.0000	0.0239	0.0026	0.0017
6.2500	0.0208	0.0024	0.0019
6.5000	0.0220	0.0023	0.0020
6.7500	0.0231	0.0022	0.0016
7.0000	0.0241	0.0020	0.0017
7.2500	0.0250	0.0020	0.0016
7.5000	0.0251	0.0019	0.0014
7.7500	0.0218	0.0018	0.0015
8.0000	0.0157	0.0017	0.0012
8.2500	0.0153	0.0016	0.0013
8.5000	0.0159	0.0015	0.0013
8.7500	0.0164	0.0015	0.0011
9.0000	0.0170	0.0014	0.0012
9.2500	0.0175	0.0013	0.0011
9.5000	0.0176	0.0013	0.0010
9.7500	0.0155	0.0013	0.0011
10.0000	0.0114	0.0012	0.0009
10.2500	0.0118	0.0012	0.0009
10.5000	0.0121	0.0011	0.0010
10.7500	0.0125	0.0011	0.0008
11.0000	0.0128	0.0011	0.0009
11.2500	0.0131	0.0010	0.0008
11.5000	0.0132	0.0010	0.0008
11.7500	0.0117	0.0010	0.0008
12.0000	0.0092	0.0009	0.0007
12.2500	0.0094	0.0009	0.0007
12.5000	0.0097	0.0009	0.0007
12.7500	0.0099	0.0008	0.0007
13.0000	0.0101	0.0008	0.0007
13.2500	0.0103	0.0008	0.0006
13.5000	0.0104	0.0008	0.0006
13.7500	0.0093	0.0007	0.0006
14.0000	0.0076	0.0007	0.0006
14.2500	0.0078	0.0007	0.0006
14.5000	0.0079	0.0007	0.0006
14.7500	0.0081	0.0007	0.0005
15.0000	0.0082	0.0007	0.0006
15.2500	0.0084	0.0006	0.0005
15.5000	0.0085	0.0006	0.0005
15.7500	0.0076	0.0006	0.0005
16.0000	0.0064	0.0006	0.0005

REQUIRED VALUES FOUND

Appendix 3·2: TABOMEGA and its subroutines

This appendix contains listings of TABOMEGA and its associated subroutines as described in §3·311, the output from the program is given in F3·6. The subroutine S17AFF called by SINCBES 1 is a function subroutine which returns the values $J_1(x)$. (IFAIL is a dummy parameter whose value is not utilised by the subroutine).

```
%EXTERNALROUTINE TABOMEGA(%STRING(63) S)
     !
     !
     !
     !      REVISION  00
     !      --------
     !
     !      PURPOSE
     !      -------  THIS ROUTINE IS USED TO TABULATE VALUES OF OMEGA C COL AS
     !      A FUNCTION OF THE NORMALISED COLLIMATOR DIAMETER 2*A/DELTA R.
     !
     !      USE
     !      ---             COMMAND:TABOMEGA(OUT)
     !
     !      OUT  IS ANY VALID OUTPUT FILE OR THE DEVICE .LP.   THE OUTPUT IS
     !           TABULATED FOR OUTPUT TO A LINE PRINTER.
     !
     !      ORIGINATOR
     !      ----------  S.H.C. HUGHES, DEPT. MEDICAL PHYSICS, R.I.E..
     !
     !
     !
     !
     %INTEGER K,I
     %REAL START,INC,FINISH,NORMDIAM
     %CONSTSTRING(43) MESSAGE="INVALID PARAMETER IN ROUTINE CALL TABOMEGA("
     !
     !
     %IF CHKFILE(S,"")=CHKFILE(S,".LP==") %AND  %C
         S#".LP" %THEN CLOSE(MESSAGE.S.")")
     START=2
     INC=0.5
     FINISH=10
     !
     K=0 ; K=K+1 %WHILE START+K*INC<FINISH
     DEFINE("ST80,".S)
     SELECTOUTPUT(80)
     SET MARGINS(80,20,132)
     NEWPAGE ; NEWLINES(15)
     PRINTSTRING("TABULATED VALUES OF F(K) = 2*PI*A*OMEGA C COL(K)")
     NEWLINES(2)
     PRINTSTRING("WHERE :- OMEGA C CCL IS THE CUT-OFF FREQ. DUE TO THE")
     NEWLINE
     PRINTSTRING("          COLLIMATOR WHEN NO DECONVOLUTION IS PERFORMED")
     NEWLINE
     PRINTSTRING("  AND    K IS THE NORMALISED DIAMETER  2*A/DELTA R")
     NEWLINE
     PRINTSTRING("----------------------------------------------------")
     NEWLINES(5)
     SET MARGINS(80,30,132)
     PRINTSTRING(" NORMDIAM       F(K)") ; NEWLINE
     PRINTSTRING(" --------       ----") ; NEWLINES(2)
     %CYCLE I=0,1,K
          NORMDIAM=START+I*INC
          PRINT(NORMDIAM,2,3)
          SPACES(5)
          PRINT(NORMDIAM*ROOT SINCBES1(NORMDIAM),2,5)
          NEWLINE
          NEWLINE %IF I&B'11'=B'11'
          %REPEAT
     NEWPAGE
     SELECTOUTPUT(0)
     CLOSESTREAM(80)
     CLEAR("ST80")
     %END
```

```
%EXTERNALLONGREALFN ROOT SINCBES1(%LONGREAL K)
    !
    !
    !      REVISION  00
    !      --------
    !
    !      PURPOSE
    !      -------  THIS ROUTINE USES THE METHED OF BISECTION TO RETURN
    !      THE SMALLEST POSITIVE ROOT OF THE EQUATION
    !
    !              SINCBES1(X,K)=1/2**1.5 =0.353 553
    !
    !      VIA %RESULT.  THE RESULT IS ACCURATE TO ABOUT 0.01%.
    !
    !
    !
    %LONGREAL B,M,T
    !
    T=3.84/K ; T=PI %IF PI<T ; B=0
    %CYCLE
         M=(B+T)/2
         %IF SINCBES1(M,K)>0.353 553 %THEN B=M %ELSE T=M
         %EXIT %IF MOD(T-B)<0.0001*B
         %REPEAT
    %RESULT=B
    %END
```

```
%EXTERNALLONGREALFN SINCBES1(%LONGREAL X,K)
    !
    !
    !
    !      REVISION  00
    !      --------
    !
    !      PURPOSE
    !      -------  THIS FN WILL RETURN THE VALUE
    !
    !              SIN(X) * J1(K*X) / (K*X*X)
    !
    !      VIA %RESULT.
    !
    !      COMMENTS
    !      --------
    !      1)   THE NAG ROUTINE S17AFF IS USED TO CALCULATE J1(K*X) .
    !
    !      ORIGINATOR
    !      ----------  S.H.C. HUGHES, DEPT. MEDICAL PHYSICS, R.I.E..
    !
    !
    !
    !
    !
    %INTEGER IFAIL
    %EXTERNALLONGREALFNSPEC S17AFF(%LONGREALNAME X,%INTEGERNAME IFAIL)
    !
    !
    %IF X=0 %THEN %RESULT=0.500000000000000
    %IF K=0 %THEN %RESULT=SIN(X)/(X+X)
    IFAIL=0
    K=K*X
    %RESULT=SIN(X)*S17AFF(K,IFAIL)/(K*X)
    %END
```

Appendix 3·3: Definition of $\omega_{c\ col}$ when deconvolution is performed.

From §2·32 E·41 $\underline{G}$ and $\underline{Q}$ are related by $\underline{G} = \sqrt{N_2}\ \underline{\underline{H}}^*\underline{Q}$. The correct inverse filter (were it not for the fact that noise is present) would be $(\sqrt{N_2}\underline{\underline{H}}^*)^{-1}$ (assuming $\underline{\underline{H}}^{*-1}$ exists), but as discussed in §2·32, the filter

$$\frac{1}{\sqrt{N_2}}\ (\underline{\underline{H}}^{*T}\underline{\underline{H}}^* + \gamma\ \underline{\underline{C}}^T\underline{\underline{C}})^{-1}\ \underline{\underline{H}}^{*T}$$

is used in practice. Thus in addition to filtering by $(\sqrt{N_2}\ \underline{\underline{H}}^*)^{-1}$ it is also necessary to filter by

$$(\underline{\underline{H}}^{*T}\underline{\underline{H}}^* + \gamma\underline{\underline{C}}^T\underline{\underline{C}})^{-1}\ \underline{\underline{H}}^{*T}\underline{\underline{H}}^*$$

because of the noise in the data.

The discrete cut-off frequency is therefore defined to be the value of k such that

$$\left(\frac{H_{k'}^* H_{k'}^*}{H_{k'}^* H_{k'}^* + \gamma C_{k'} C_{k'}}\right)^2 \geq \frac{1}{2} \quad \forall\, 0 \leq k' \leq k \quad \text{and}$$

$$\left(\frac{H_{k+1}^* H_{k+1}^*}{H_{k+1}^* H_{k+1}^* + \gamma C_{k+1} C_{k+1}}\right)^2 < \frac{1}{2}$$

i.e. such that

$$H_k^*(\sqrt{2} - 1)^{\frac{1}{2}} \geq \sqrt{\gamma}\ C_k \quad \forall \quad 0 \leq k' \leq k \quad \text{and} \quad H_{k+1}^*(\sqrt{2} - 1)^{\frac{1}{2}} < \sqrt{\gamma}\ C_{k+1}$$

(since C_k and H_k^* are real).

This k corresponds to the component of frequency $k\Delta R = k(\Delta r N_2)^{-1}$ in continuous frequency space and the cut-off frequency $\omega_{c\ col}$ is therefore defined by

$$\omega_{c\ col} = k(\Delta r N_2)^{-1}$$

with k defined as above.

Appendix 3·4 Calculation of p from f.

·1 Summary of methods of calculating p.

Given the phantom f there are three natural methods by which p could be calculated:-

Method I : Apply the F.F.T. to the array f_{jk} to obtain an estimate of the function F and then calculate p from this.

Method II: In order to avoid the computation of the two dimensional F.F.T. inherent in I, one could develop an analytic expression for F by utilising the special structure of f given in §3·4311 and then evaluate this at selected sample points. One could then evaluate p as before.

Method III: Calculate p direct from f by simply evaluating the line integrals and then evaluate F by using the F.F.T. on p.

Each of these methods is now examined in turn. In order to develop some feel for the size of the numerical problems involved, the case $N_4 = 100$ and $M = 40$ will be considered.

·2 Method I

On the particular computer used and with the library routines available this is not practicle. With $N_4 = 100$ then the phantom array f_{jk} requires $100 \times 100 \times 8$ bytes of storage (the F.F.T. algorithms required REAL*8 variables), further it would have been necessary to furnish the routines with two arrays of this size (for the real and complex parts of F) thus if the original data was not to be destroyed a total of

$$3 \times 100 \times 100 \times 8 \text{ bytes} \simeq \frac{1}{4} \text{ M byte}$$

would have been required for major arrays alone. This was not acceptable and the method was considered no further.

·3 Method II

In order to avoid the computational requirements of Method I, analytic expressions for F in terms of f_{jk} can be developed.

Consider a phantom with one non-zero cell defined by

$$f_1(x,y) = \begin{cases} 1 & (x,y) \in \left(x_0 - \frac{\Delta r_2}{2},\ x_0 + \frac{\Delta r_2}{2} \right) \times \left(y_0 - \frac{\Delta r_2}{2},\ y_0 + \frac{\Delta r_2}{2} \right) \\ 0 & \text{Otherwise} \end{cases}$$

then $F_1(X,Y) = \mathcal{F}_2\, f_1(x,y)$

$$= \Delta r_2^2\, e^{-2\pi i(x_0 X + y_0 Y)} \operatorname{sinc}(\Delta r_2 X) \operatorname{sinc}(\Delta r_2 Y)$$

where $\operatorname{sinc}(x) = \dfrac{\sin \pi x}{\pi x}$

and $F_1(R,\phi) = \Delta r_2^2\, e^{-2\pi i R(x_0 \cos\phi + y_0 \sin\phi)} \operatorname{sinc}(\Delta r_2 R \cos\phi) \operatorname{sinc}(\Delta r_2 R \sin\phi)$.

Since f is the sum of $N_4 \times N_4$ such functions whose centres (x_0, y_0) are given by §3·4311 E·21 then

$$\begin{aligned} F(R,\phi) &= \mathcal{F}_2 f(x,y) \\ &= \Delta r_2^2 \sum_{j,k=1}^{N_4} f_{jk} \exp\left\{-2\pi i R\left[\left(j - \frac{N_4+1}{2}\right) r_2 \cos\phi + \left(k - \frac{N_4+1}{2}\right) r_2 \sin\phi\right]\right\} \\ &\qquad \times \operatorname{sinc}(\Delta r_2 R \cos\phi) \operatorname{sinc}(\Delta r_2 R \sin\phi) \end{aligned}$$

Now consider the same numerical example. With $N_4 = 100$ this summation contains 10^4 terms. Further, since $\Delta r' \le \Delta r_2/4$ (§3·4312), each projection will contain at least 400 sample points and with $M = 40$ this means F must be calculated at $40 \times 400 = 16 \times 10^3$ points. The evaluation of a single term in the summation requires the calculation of four sine or cosine functions, so that the evaluation of F at all the required sample points means calculating

$$10^4 \times (16 \times 10^3) \times 4 \simeq 6{\cdot}4 \times 10^8$$

trigonometric functions. This was also considered unacceptable and the method taken no further.

·4 Method III

In order to avoid the large core requirements of Method I and the large number of trigonometric evaluations of Method II, a procedure is now given for the direct evaluation of p as the line integral of f, from this the function F may be evaluated by use of the F.F.T..

·41 Definition of r_{min}, r_{mid} and r_{max}

Consider the projection of the phantom, in particular the contribution of the (j,k)th cell of the phantom to a projection at some angle ϕ (see F·1a). It is clear that this cell will only make a contribution to $p(r,\phi)$ in the range

$$r_{min} \leq r \leq r_{max}$$

and in the following sections expressions are developed for r_{min}, r_{max} and the contribution made between them.

·411 Determination of $r_{max} - r_{min}$

There are two possibilities to be considered: $0 \leq \phi \leq \pi/2$ and $\pi/2 \leq \phi \leq \pi$ In the first case (see F·1b).

$$r_{max} - r_{min} = \sqrt{2}\Delta r_2 \cos\left(\frac{\pi}{4} - \phi\right)$$

where the phantom element has sides parallel to the axes and of length Δr_2. In the second case (see F·1c)

$$r_{max} - r_{min} = \sqrt{2}\,\Delta r_2 \cos\left((\pi - \phi) - \frac{\pi}{4}\right) .$$

These results may be combined to give

$$r_{max} - r_{min} = \Delta r_2(|c| + |s|) \qquad 0 \leq \phi \leq \pi \qquad \cdot 1$$

where $c = \cos\phi$ and $s = \sin\phi$.

(a)

(b)

(c)

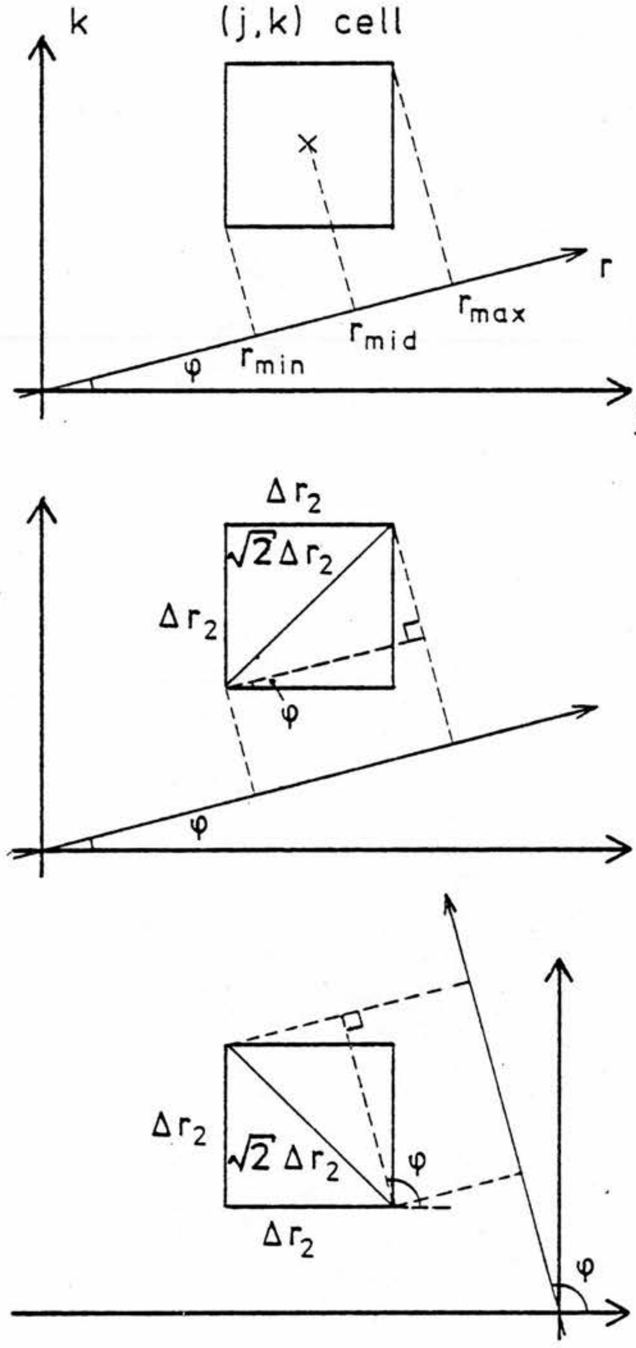

k
(j,k) cell
r
r_{max}
r_{mid}
r_{min}
φ
j
Δr_2
$\sqrt{2}\Delta r_2$
Δr_2
φ
φ
Δr_2
$\sqrt{2}\Delta r_2$
φ
Δr_2
φ

fig. 3·4·1.

·412 Determination of r_{mid}

If the phantom consists of an $N_4 \times N_4$ matrix of cell (indexing from 1 to N_4) symmetrically placed about the axes then the coordinates of the centre of the (j,k)th cell are

$$\left(\left(j - \frac{N_4 + 1}{2}\right)\Delta r_2,\quad \left(k - \frac{N_4 + 1}{2}\right)\Delta r_2\right)$$

(see §3·4311). Thus by taking the component of this vector in the direction (cos ϕ, sin ϕ)

$$r_{mid} = \left(j - \frac{N_4 + 1}{2}\right) \Delta r_2 c + \left(k - \frac{N_4 + 1}{2}\right) \Delta r_2 s \qquad \cdot 2$$

·413 Determination of r_{max} and r_{min}

Combining E·1, E·2 and the fact that r_{max} and r_{min} are symmetrically placed about r_{mid} gives

$$r_{max} = \left(j - \frac{N_4 + 1}{2}\right) \Delta r_2 c + \left(k - \frac{N_4 + 1}{2}\right) \Delta r_2 s + \Delta r_2 \frac{|c| + |s|}{2} \qquad \cdot 3$$

$$r_{min} = \left(j - \frac{N_4 + 1}{2}\right) \Delta r_2 c + \left(k - \frac{N_4 + 1}{2}\right) \Delta r_2 s - \Delta r_2 \frac{|c| + |s|}{2}$$

·42 Determination of cells in projection to which (j,k)th element of phantom contributes.

If $p(r,\phi)$ is sampled at N_3 points (indexed 0 to $N_3 - 1$) symmetrically placed about the origin with sample spacing $\Delta r'$ then

$$r_\ell = \left(\ell - \frac{N_3 - 1}{2}\right) \Delta r'$$

and the samples affected by the (j,k)th element of f are those for which

$$r_{min} \le r_\ell \le r_{max}$$

i.e. those for which

$$\frac{\Delta r_2}{\Delta r'}\left\{\left(j-\frac{N_4+1}{2}\right)c+\left(k-\frac{N_4+1}{2}\right)s-\frac{|c|+|s|}{2}\right\}+\frac{N_3-1}{2} \tag{·4}$$

$$\leq \ell \leq \frac{\Delta r_2}{\Delta r'}\left\{\left(j-\frac{N_4+1}{2}\right)c+\left(k-\frac{N_4+1}{2}\right)s+\frac{|c|+|s|}{2}\right\}+\frac{N_3-1}{2}$$

(Given $\phi, \Delta r_2, \Delta r', N_3$ and N_4 the set of all (j,k) such that

(1) $1 \leq j,k \leq N_4$

and (2) E·4 is true

will be denoted by $A(\ell,\phi,\Delta r_2,\Delta r',N_3,N_4)$ or sometimes $A(\ell)$.)

·43 Determination of line integral of a single cell

Consider the determination of the line integrals of a phantom consisting of a single cell of unit weight with side of length Δr_2 and centred at the origin.

·431 Expressions for different ϕ

·4311 Case 1: $0 \leq \phi \leq \pi/4$

The coordinates of any point on the line L (see F·2) can be written

$(rc,rs) + \tau(-s,c)$

(where τ is some constant, $c = \cos\phi$ and $s = \sin\phi$) and the line integral of this phantom along L is simply the length AB.

The point B (determined by $\tau = \tau_1$ say) is given by

$\tau_1 : rc - \tau_1 s = 4r_2/2$

since the x-coordinate of B is $\Delta r_2/2$. Similarly the point A is determined by

$\tau_2 : rs + \tau_2 c = \Delta r_2/2$

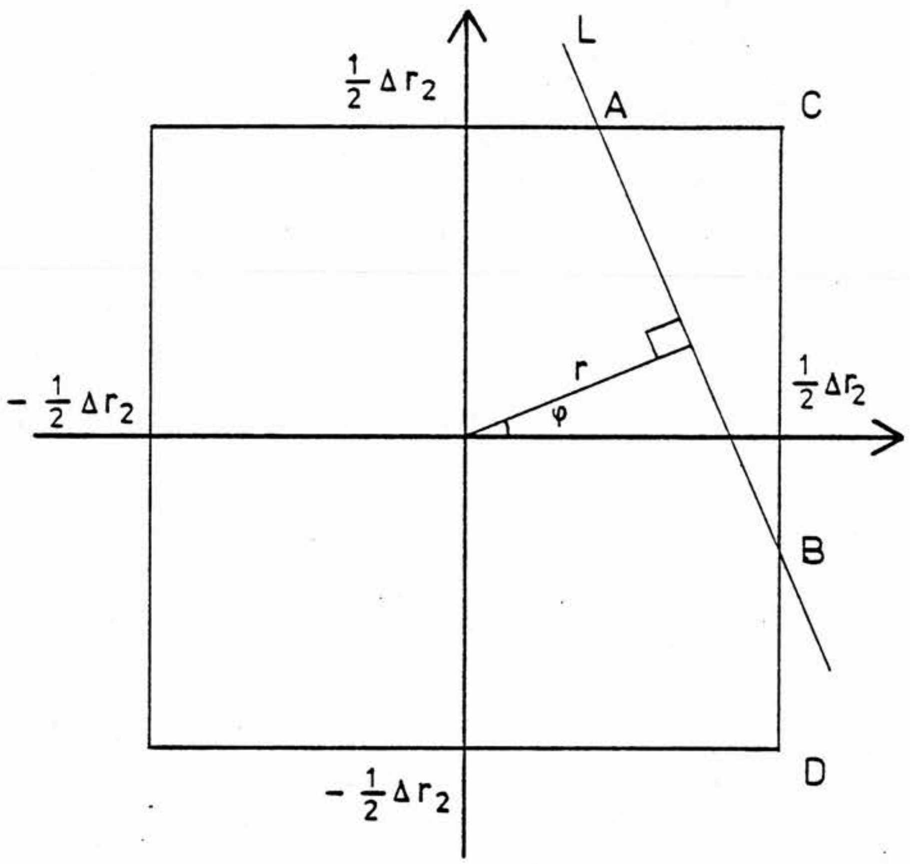

L
A
C
$\frac{1}{2}\Delta r_2$
r
φ
$-\frac{1}{2}\Delta r_2$
$\frac{1}{2}\Delta r_2$
B
D
$-\frac{1}{2}\Delta r_2$

fig. 3·4·2.

and $$AB = |\tau_1 - \tau_2|$$
$$= \left| \frac{rc - \frac{\Delta r_2}{2}}{s} + \frac{rs - \frac{\Delta r_2}{2}}{c} \right|$$
$$= \left| \frac{r - \Delta r_2 \frac{c + s}{2}}{cs} \right|$$
$$= \frac{\Delta r_2}{|c|\ |s|} \left| \frac{c + s}{2} - \frac{r}{\Delta r_2} \right|$$

This formula remains valid provided L intersects CD. Now the value of r when L passes through C is given by

$$rc - \tau s = \frac{\Delta r_2}{2} = rs + \tau c$$

since C has coordinates $(\Delta r_2/2,\ \Delta r_2/2)$, thus $r = \Delta r_2 \left(\frac{c + s}{2} \right)$.

Similarily for D

$$rc - \tau s = \frac{\Delta r_2}{2} = -(rs + \tau c)$$

hence

$$r = \Delta r_2 \left(\frac{c - s}{2}\right)$$

and the formula for AB is seen to be valid if

$$\frac{c + s}{2} \geq \frac{r}{\Delta r_2} \geq \frac{c - s}{2}$$

If $$\frac{c - s}{2} \geq \frac{r}{\Delta r_2} \geq 0$$

(i.e. L passes between the origin and D) then the line integral has value

$$\frac{\Delta r_2}{|c|\ |s|} \left| \frac{c + s}{2} - \frac{c - s}{2} \right|$$

by substituting $\frac{r}{\Delta r_2} = \frac{c - s}{2}$.

Summarising then:- the line integral has the value

$$\frac{\Delta r_2}{|c|\ |s|} \left| \frac{c + s}{2} - \frac{r}{\Delta r_2} \right| \qquad \frac{c + s}{2} \geq \frac{r}{\Delta r_2} \geq \frac{c - s}{2}$$

$$\frac{\Delta r_2}{|c|\ |s|} \left| \frac{c + s}{2} - \frac{c - s}{2} \right| \qquad \frac{c - s}{2} \geq \frac{r}{\Delta r_2} \geq 0$$

·4312 Case 2: $\pi/4 \le \phi \le \pi/2$

Precisely similar reasoning gives

$$\frac{\Delta r_2}{|c|\ |s|}\left|\frac{c+s}{2} - \frac{r}{\Delta r_2}\right| \qquad \frac{c+s}{2} \ge \frac{r}{\Delta r_2} \ge -\frac{c-s}{2}$$

$$\frac{\Delta r_2}{|c|\ |s|}\left|\frac{c+s}{2} + \frac{c-s}{2}\right| \qquad -\frac{c-s}{2} \ge \frac{r}{\Delta r_2} \ge 0$$

·4313 Case 3: $\pi/2 \le \phi \le 3\pi/4$

$$\frac{\Delta r_2}{|c|\ |s|}\left|\frac{s-c}{2} - \frac{r}{\Delta r_2}\right| \qquad \frac{s-c}{2} \ge \frac{r}{\Delta r_2} \ge \frac{c+s}{2}$$

$$\frac{\Delta r_2}{|c|\ |s|}\left|\frac{s-c}{2} - \frac{c+s}{2}\right| \qquad \frac{c+s}{2} \ge \frac{r}{\Delta r_2} \ge 0$$

·4314 Case 4: $3\pi/4 \le \phi \le \pi$

$$\frac{\Delta r_2}{|c|\ |s|}\left|\frac{s-c}{2} - \frac{r}{\Delta r_2}\right| \qquad \frac{s-c}{2} \ge \frac{r}{\Delta r_2} \ge -\frac{c+s}{2}$$

$$\frac{\Delta r_2}{|c|\ |s|}\left|\frac{s-c}{2} + \frac{c+s}{2}\right| \qquad -\frac{c+s}{2} \ge \frac{r}{\Delta r_2} \ge 0$$

·432 Expression for $0 \le \phi \le \pi$

The four upper limits

$$\frac{c+s}{2},\ \frac{c+s}{2},\ \frac{s-c}{2} \text{ and } \frac{s-c}{2}$$

for the four cases above can be combined in the single expression

$$\frac{|c| + |s|}{2} \quad \text{for } 0 \le \phi \le \pi$$

and the four lower limits

$$\frac{c-s}{2}\ ,\ -\frac{c-s}{2}\ ,\ \frac{c+s}{2}\ \text{and}\ -\frac{c+s}{2}$$

in the expression

$$\left|\frac{|c| - |s|}{2}\right|$$

Thus the four expressions for the line integrals can be combined to

$$\frac{\Delta r_2}{|c||s|}\left\{\frac{|c|+|s|}{2} - \frac{r}{\Delta r_2}\right\} \qquad \frac{|c|+|s|}{2} \geq \frac{r}{\Delta r_2} \geq \left|\frac{|c|-|s|}{2}\right|$$

$$\frac{\Delta r_2}{|c||s|}\left\{\frac{|c|+|s|}{2} - \left|\frac{|c|-|s|}{2}\right|\right\} \qquad \left|\frac{|c|-|s|}{2}\right| \geq \frac{r}{\Delta r_2} \geq 0$$

further it is clear that the line integral will have the same value for $\pm$ r thus giving

$$\frac{\Delta r_2}{|c||s|}\left\{\frac{|c|+|s|}{2} - \left|\frac{r}{\Delta r_2}\right|\right\} \qquad \frac{|c|+|s|}{2} \geq \left|\frac{r}{\Delta r_2}\right| \geq \left|\frac{|c|-|s|}{2}\right|$$

$$\frac{\Delta r_2}{|c||s|}\left\{\frac{|c|+|s|}{2} - \left|\frac{|c|-|s|}{2}\right|\right\} \qquad \left|\frac{|c|-|s|}{2}\right| \geq \left|\frac{r}{\Delta r_2}\right| \geq 0\ . \qquad \cdot 5$$

This function of $r/\Delta r_2$ is denoted by $p^*\left(\frac{r}{\Delta r_2}, \Delta r_2, \phi\right)$ or $p^*(r/\Delta r_2)$ for short. It will be found unnecessary to define p^* for other values of $r/\Delta r_2$.

·44 Determination of contribution of (j,k)th element of phantom to ℓth cell of projection

The position of the ℓth sample is given by

$$r_\ell = \left(\ell - \frac{N_3 - 1}{2}\right)\Delta r'$$

(see §·42) and the midpoint of the (j,k)th element projects onto

$$r_{mid} = \left(j - \frac{N_4+1}{2}\right) \Delta r_2 c + \left(k - \frac{N_4+1}{2}\right) \Delta r_2 s$$

(see §·412) thus the distance of r_ℓ from r_{mid} in units of Δr_2 is

$$\begin{aligned} \text{dev} &= \frac{1}{\Delta r_2} \left| r_{mid} - r_\ell \right| \\ &= \left| \left(j - \frac{N_4+1}{2}\right) c + \left(k - \frac{N_4+1}{2}\right) s + \frac{N_3-1}{2} \frac{\Delta r'}{\Delta r_2} - \ell \frac{\Delta r'}{\Delta r_2} \right| \\ &= \frac{\Delta r'}{\Delta r_2} \left| \left(j - \frac{N_4+1}{2}\right) \frac{\Delta r_2}{\Delta r'} c + \left(k - \frac{N_4+1}{2}\right) \frac{\Delta r_2}{\Delta r'} s + \frac{N_3-1}{2} - \ell \right| \end{aligned} \qquad \cdot 6$$

Thus for those j,k such that $(j,k) \in A(\ell)$ the contribution of f_{jk} to $(\underline{p})_\ell$ is given by substituting dev in place of $|r/\Delta r_2|$ in E·5 and multiplying by f_{jk} (the elements f_{jk} where $(j,k) \notin A(\ell)$ contribute nothing).

·45 Complete expression for line integral of phantom

Combining the results of §§·42, ·432 and ·44 gives

$$\left(\underline{p}(\phi)\right)_\ell = \sum_{(j,k)\in A(\ell)} f_{jk}\, p^*\left(\text{dev}(j,k,\ell,\phi)\right)$$

where $A(\ell)$ is defined in §·42

p^* is defined in §·432

and dev is defined in §·44

Now consider how this relates to Methods I and II when $N_4 = 100$ and $M = 40$. The first point to note is that the summation is

$$\sum_{(j,k)\in A(\ell)} \quad \text{rather than} \quad \sum_{1\le j,k\le N_4}$$

It is easy to see that $A(\ell)$ contains less than 200 points (consider a line of integration parallel to $\phi = \rho/4$ but just slightly offset from the origin - it will intersect only the 100 diagonal elements plus the 99 elements just off the diagonal), it is clearly preferable to sum over less than 200 points rather than sum contributions over 10^4 points most of which contribute nothing.

The computational strategy for evaluating $\underline{p}(\phi)$ should be:-

1) set up arrays for f_{jk} and $\underline{p}(\phi)$
2) choose ϕ and calculate c and s
3) start with $f_{1,1}$ and add its contribution to each of the relevant elements of $\underline{p}(\phi)$ (see §·42)
4) repeat (3) with $f_{1,2}$
5) repeat (3) with $f_{1,3}$

$\vdots$

repeat (3) with f_{N_4,N_4}

There are several important consequences of such a strategy:-

a) space is required for one $N_4 \times N_4$ array and for one N_3 array; compared with Method I this is a significant improvement.
b) it is only necessary to calculate two trigonometric functions for each projection - a total of 80 (c.f. Method II)
c) each element of f_{jk} is accessed only once when calculating each projection and the elements are accessed in order (this is significant when using a computer with an operating system which uses virtual memory as large numbers of unnecessary page turns must be avoided).

The software used to implement these ideas may be found in Appendix 3·5.

Appendix 3·5: Prediction software

This appendix contains the software discussed in §3·432. The software includes the mainline PREDICT, plus four associated subroutines and brief notes on one library routine.

```
!
!     CONSTANTS
!
%CONSTINTEGER REVERSE=0,FORWARD=1
%CONSTREAL KLAMMAX=2.1877@-3
%CONSTLONGREAL BCELLEN=0.020320000000000
%CONSTLONGREAL BEAM DENSITY=5.4685@5  ;! QUANTA / SEC. CMS**2
%EXTERNALROUTINE PREDICT(%STRING(63) S)
   !
   !
   !
   !     REVISION  00
   !     --------
   !
   !     PURPOSE
   !     -------   THIS ROUTINE IS FOR INTERACTIVE USE ONLY AND IS USED TO
   !     PREDICT HOW A SECTION SHOULD BE SCANNED IF A STATED QUALITY OF
   !     RESULT IS TO BE OBTAINED.
   !
   !     USE
   !     ---            COMMAND:PREDICT(INFILE,OUT)
   !
   !     INFILE     IS THE DATA SET REFERENCE NUMBER.
   !     OUT        IS ANY VALID OUTPUT FILE OR THE DEVICES .TT OR .LP.
   !
   !     COMMENTS
   !     --------
   !     1)   IF Z====..PPREDICT DOES NOT EXIST IT IS CREATED BY THE ROUTINE
   !          AND ALL THE PARAMETERS CONTAINED IN IT ARE CALCULATED. IF
   !          Z====..PPREDICT ALREADY EXISTS THEN THE ROUTINE MERELY LISTS
   !          ITS CONTENTS TO OUT.
   !     2)   THE ROUTINE MAKES TEMPORARY USE OF I/O CHANNELS 78,79,80.
   !
   !     ORIGINATOR
   !     ----------  S.H.C. HUGHES, DEPT. MEDICAL PHYSICS, R.I.E..
   !
   !
   !
   !
   !
   %CONSTSTRING(2) REV="00"
   %CONSTSTRING(39) MESSAGE="INVALID PARAMETER IN ROUTINE   PREDICT("
   %STRING(63) %ARRAY T(1:3)
   %RECORDNAME PPRED(PPREDFORM)
   %ROUTINESPEC READ IN PARAMETERS(%RECORDNAME PPRED)
   %ROUTINESPEC PREDICT PARAMETERS(%RECORDNAME PPRED)
   %ROUTINESPEC CALC SCANNER SETTINGS(%RECORDNAME PPRED)
   !
   !
```

```
!
DECOMP(S,T,2)
%IF CHKFILE(T(1),"Z====")="BAD"  %OR %C
      (T(2)#"" %AND CHKFILE(T(2),"")=CHKFILE(T(2),".LP==") %AND  %C
      ".LP"#T(2)#".TT") %THEN CLOSE(MESSAGE.S.")")
T(2)=".TT" %IF T(2)=""
!
!
%IF EXIST(T(1)."_PPREDICT")=0 %THEN %START
      DESTROY("ZZ#TEMP") %IF EXIST("ZZ#TEMP")#0
      PPRED==RECORD(ADSMFILE("ZZ#TEMP",80,RECLENGTH("PPRED")))
      %FINISH %ELSE %START
      PPRED==RECORD(ADSMFILE(T(1)."_PPREDICT",80,0))
      -> LIST
      %FINISH
!
!
PPRED_REV=REV
READ IN PARAMETERS(PPRED)
PREDICT PARAMETERS(PPRED)
CALC SCANNER SETTINGS(PPRED) %IF 1<=PPRED_OPTION<=5
LIST:
DEFINE("ST79,".T(2))
SELECTOUTPUT(79)
%IF T(2)->(".LP").T(2) %START
      SET MARGINS(79,30,130)
      NEWPAGE
      NEWLINES(10)
      T(2)=".LP".T(2)
      %FINISH
T(1)->("Z").T(1)
LPPRED(PPRED,"PRED",T(1))
SELECTOUTPUT(0)
CLOSESTREAM(79)
CLOSESM(80)
CLEAR("ST79,SM80")
%IF EXIST("Z".T(1)."_PPREDICT")=0 %START
      CLOSESM(78)
      CLEAR("SM78")
      COPYFILE("ZZ#TEMP,Z".T(1)."_PPREDICT")
      DESTROY("ZZ#TEMP")
      %FINISH
!
!
!
%ROUTINE READ IN PARAMETERS(%RECORDNAME PPRED)
     !
     !
     %SWITCH SWPRED(1:6)
```

```
%RECORDSPEC PPRED(PPREDFORM)
!
!
NEWLINES(3)
PRINTSTRING("SET PARAMETERS FOR PROGRAM PREDICT") ; NEWLINE
PRINTSTRING("---------------------------------") ; NEWLINE
PRINTSTRING("---------------------------------") ; NEWLINES(2)
!
!     READ  OMEGA C REC AND CALC DCOLDIAM
!
NEWLINES(3)
PRINTSTRING("CHOOSE METHOD OF ASSIGNING OMEGA C REC") ; NEWLINE
PRINTSTRING("--------------------------------------") ; NEWLINES(2)
PRINTSTRING("     1 : OMEGA C REC DETERMINED BY 1MM COLIMATOR")
NEWLINE
PRINTSTRING("     2 : OMEGA C REC DETERMINED BY 2MM COLIMATOR")
NEWLINE
PRINTSTRING("     3 : OMEGA C REC DETERMINED BY 3MM COLIMATOR")
NEWLINE
PRINTSTRING("     4 : OMEGA C REC DETERMINED BY 5MM COLIMATOR")
NEWLINE
PRINTSTRING("     5 : OMEGA C REC DETERMINED BY 10MM COLIMATOR")
NEWLINE
PRINTSTRING("     6 : OMEGA C REC GIVEN BY USER (UNITS=CM**-1)")
NEWLINES(2)
PROMPT("OPTION? : ")
READ(PPRED_OPTION) %UNTIL 1<=PPRED_OPTION<=6
->SWPRED(PPRED_OPTION)
!
!
SWPRED(1):PPRED_DCOLDIAM=0.1000 ; ->OMEGA
SWPRED(2):PPRED_DCOLDIAM=0.2000 ; ->OMEGA
SWPRED(3):PPRED_DCOLDIAM=0.3000 ; ->OMEGA
SWPRED(4):PPRED_DCOLDIAM=0.5000 ; ->OMEGA
SWPRED(5):PPRED_DCOLDIAM=1.0000 ; ->OMEGA
OMEGA:PPRED_OMEGACREC=1/(2*PPRED_DCOLDIAM)
      -> LINE
SWPRED(6):!USER SUPPLIED VALUE
          PROMPT("OMEGA C REC? : ")
          READ(PPRED_OMEGACREC) %UNTIL PPRED_OMEGACREC>0
          PPRED_DCOLDIAM=1/(2*PPRED_OMEGACREC)
!
!     READ IN CAPL
!
LINE:
PROMPT("DATA:")
NEWLINE
PRINTSTRING("GIVE LINE LENGTH (IN CMS) TO BE USED IN SCAN") ; NEWLINE
READ(PPRED_CAPL) %UNTIL PPRED_CAPL>0
```

```
    !
    !      READ IN THETA
    !
    THETA:
    NEWLINE
    PRINTSTRING("GIVE MAX. ALIASING ERROR (IN %) ALLOWED IN M-DATA")
    NEWLINE
    READ(PPRED_THETA) ; PPRED_THETA=PPRED_THETA/100
    %IF PPRED_THETA<0.008 %START
         PRINTSTRING("ERROR TOO SMALL") ; NEWLINE
         ->THETA
         %FINISH
    %IF PPRED_THETA>0.10 %START
         PRINTSTRING("ERROR TOO LARGE") ; NEWLINE
         ->THETA
         %FINISH
    !
    !      READ IN LITM
    !
    NEWLINE
    PRINTSTRING("GIVE MIN. NO OF SAMPLES IN FIRST PEAK OF S")
    NEWLINE
    READ(PPRED_LITM) %UNTIL PPRED_LITM>=2
    %END

!
!
!
%ROUTINE PREDICT PARAMETERS(%RECORDNAME PPRED)
    !
    !
    !      PURPOSE
    !      -------  THIS ROUTINE CALCULATES THE PARAMETERS CAPD,LITDELR,
    !      CAPM,LITL.
    !
    !
    %RECORDSPEC PPRED(PPREDFORM)
    %INTEGER ICONST,J,K,N3,LINE,CELL,M1,N4
    %LONGREAL TEMPRL,TEMPRL1,LITDELRD,LAMDAD,S1,S2,LITDELR2
    %RECORDNAME PPHANTOM(PPHANTOMFORM)
    !
    !      READ IN PARAMETERS
    !
    PPHANTOM==RECORD(ADSMFILE(T(1)."_PPHANTOM",78,0))
    LITDELR2=PPHANTOM_LITDELRP
    N4=PPHANTOM_NP
    CLOSESM(78)
    CLEAR("SM78")
    !
```

```
!     SET UP PHANTOM DATA ARRAY
!
%BEGIN
    %LONGREALARRAYNAME LITF
    %LONGREALARRAYFORMAT LITFFORM(1:N4,1:N4)
    LITF==ARRAY(ADSMFILE(T(1)."_PHANTOM",78,N4*N4*8),LITFFORM)
    !
    !     CALC DIAM OF PHANTOM
    !
    PPRED_CAPD=0 ; TEMPRL=(N4+1)/2
    %CYCLE J=1,1,N4
         %CYCLE K=1,1,N4
              TEMPRL1=(J-TEMPRL)**2+(K-TEMPRL)**2
              PPRED_CAPD=TEMPRL1      %C
                   %IF PPRED_CAPD<TEMPRL1 %AND LITF(J,K)>1@-20
              %REPEAT
         %REPEAT
    PPRED_CAPD=LITDELR2*SQRT(PPRED_CAPD)
    !
    !     CALC LITDELR
    !
    PPRED_LITDELR=EXP(LOG(KLAMMAX*(SQRT(2)-1))/5)/      %C
                        (PPRED_OMEGACREC*(PPRED_LITM-1))
    TEMPRL=EXP(2*LOG(2*PPRED_THETA)/3)*PPRED_DCOLDIAM
    PPRED_LITDELR=TEMPRL %IF TEMPRL<PPRED_LITDELR
    !
    !     CALC CAPM
    !
    PPRED_CAPM=INT(PI*PPRED_OMEGACREC*PPRED_CAPD+1)
    !
    !     CALCULATE L
    !
    !     CALC LITDELRD
    !
    LITDELRD=KLAMMAX*EXP(LOG(SQRT(2)-1)/5)/     %C
                  (PPRED_OMEGACREC*(PPRED_LITM-1))
    TEMPRL=LITDELR2/4
    LITDELRD=TEMPRL %IF LITDELRD<TEMPRL
    !
    !      CALC. N3 AND M1
    !
    TEMPRL=(SQRT(2)-1)*EXP(LOG(2000)/5)/PPRED_OMEGACREC
         ! IF THETA=0.01 THEN N LAMDA D MIN(THETA)=2000
    TEMPRL=PPRED_CAPL %IF TEMPRL<PPRED_CAPL
    TEMPRL=TEMPRL/LITDELRD
    N3=RADIX2(INT(TEMPRL+1))
    ICONST=N3 ; M1=0
    ICONST=ICONST>>1 %AND M1=M1+1 %UNTIL ICONST=0
    !
```

```
!      SET UP DATA ARRAYS
!
%BEGIN
     %INTEGER MIN,MAX
     %LONGREAL RATIO,N3D,N4D,C,CC,S,SS,SUM,DIFF,SUMRAT,MID,MIDD,DEV,ANGLE
     %LONGREALARRAY RELINE,IMLINE,CAPLAM5(0:N3-1),WORK(1:M1)
     %LONGREALARRAYNAME CAPS
     CAPS==ARRAY(ADDR(CAPLAM5(0)),CAPLAM5)
     !
     !      CALC CAPS,RATIO,N3D,N4D
     !
     LAMDAD=(LITDELRD*N3*PPRED_OMEGACREC)**5/(SQRT(2)-1)
     %CYCLE CELL=0,1,N3-1
          CAPLAM5(CELL)=CAPLAMDA(CELL,N3)**5
          CAPS(CELL)=CAPLAM5(CELL)/(CAPLAM5(CELL)+LAMDAD)
          %REPEAT
     RATIO=LITDELR2/LITDELRD
     N3D=(N3-1)/2
     N4D=(N4+1)/2
     !
     !      CALC. RELINE,IMLINE
     !
     S1=0 ; S2=0
     %CYCLE LINE=1,1,PPRED_CAPM
          ANGLE=PI*(LINE-1)/PPRED_CAPM
          C=COS(ANGLE)
          S=SIN(ANGLE)
          SUM=(MOD(C)+MOD(S))/2
          DIFF=MOD(MOD(C)-MOD(S))/2
          SUMRAT=RATIO*SUM
          CC=RATIO*C
          SS=RATIO*S
          MIDD=N3D-N4D*(CC+SS)
          RLMATO(RELINE,0,N3-1)
          RLMATO(IMLINE,0,N3-1)
          %CYCLE K=1,1,N4
               MIDD=MIDD+SS
               MID=MIDD
               %CYCLE J=1,1,N4
                    MID=MID+CC
                    MIN=-INT PT(SUMRAT-MID)
                    MAX=INT PT(MID+SUMRAT)
                    MIN=0 %IF MIN<0
                    MAX=N3-1 %IF MAX>N3-1
                    ->LABEL %IF MIN>MAX
                    %CYCLE CELL=MIN,1,MAX
                         DEV=MOD(MID-CELL)/RATIO
                         RELINE(CELL)=RELINE(CELL)+(SUM-DEV)*LITF(J,K)     %C
                              %IF SUM>DEV>DIFF
```

```
                        %IF DIFF>=DEV %START
                            %IF C#0#S %THEN RELINE(CELL)=RELINE(CELL)+ %C
                                (SUM-DIFF)*LITF(J,K) %C
                                %ELSE RELINE(CELL)=RELINE(CELL)+LITF(J,K)
                            %FINISH
                        %REPEAT
                    LABEL:%REPEAT
                %REPEAT
            TEMPRL1=LITDELR2
            TEMPRL1=TEMPRL1/S %IF S#0
            TEMPRL1=TEMPRL1/C %IF C#0
            %CYCLE CELL=0,1,N3-1
                RELINE(CELL)=RELINE(CELL)*TEMPRL1
                %REPEAT
        !
        !     CALC S1
        !
            TEMPRL1=0
            %CYCLE CELL=0,1,N3-1
                TEMPRL1=TEMPRL1+EXP(RELINE(CELL))    %C
                    %IF MOD((CELL-(N3-1)/2)*LITDELRD)<=PPRED_CAPL/2
                %REPEAT
            S1=S1+TEMPRL1
        !
        !     CALC S2
        !
            C06ABF(RELINE(0),IMLINE(0),N3,FORWARD,M1,WORK(1))
            TEMPRL1=0
            %CYCLE CELL=0,1,N3-1
                TEMPRL1=TEMPRL1+CAPS(CELL)**2*     %C
                        (RELINE(CELL)**2+IMLINE(CELL)**2)
                %REPEAT
            S2=S2+TEMPRL1
            %REPEAT
        !
        !   CALC LITL
        !
        PPRED_LITL=S1/S2
        %END
    %END
  !
  !
  !
  %END
!
!
```

```
!
%ROUTINE CALC SCANNER SETTINGS(%RECORDNAME PPRED)
    !
    !
    %INTEGER I
    %LONGREAL TEMPRL
    %RECORDSPEC PPRED(PPREDFORM)
    !
    !
    PPRED_SMPF=PPRED_DCOLDIAM**2*PPRED_LITDELR*4*BEAM DENSITY/  %C
                   (PPRED_LITL*BCELLEN)
    I=0 ; TEMPRL=PPRED_SMPF
    TEMPRL=TEMPRL/10 %AND I=I+1 %WHILE TEMPRL>=10
    PPRED_RNDSMPF=5*10**I %IF 10>TEMPRL>=5
    PPRED_RNDSMPF=2*10**I %IF 5>TEMPRL>=2
    PPRED_RNDSMPF=10**I %IF 2>TEMPRL>=1
    !
    PPRED_GPSF=PPRED_LITDELR/BCELLEN
    PPRED_RNDGPSF=INT PT(PPRED_GPSF)
    PPRED_RNDGPSF=RADIX2(PPRED_RNDGPSF)>>1 %IF 0#BITS(PPRED_RNDGPSF)#1
    PPRED_BLANKLINE=INT(36*PPRED_LITDELR/   %C
                       (PPRED_CAPL*PPRED_LITL*PPRED_THETA**2)+1/2)
    PPRED_RNDK=PPRED_DCOLDIAM/(PPRED_RNDGPSF*BCELLEN)
    PPRED_RNDTHETA=EXP(LOG(PPRED_RNDK)*(-1.5000000))/2
    PPRED_RNDLITM=2*EXP(LOG((SQRT(2)-1)*KLAMMAX)/5)*PPRED_RNDK+1
    PPRED_RNDLITL=4*PPRED_DCOLDIAM**3*BEAM DENSITY/  %C
                (PPRED_RNDK*PPRED_RNDSMPF*BCELLEN)
    %END
!
!
!
%END
```

```
%EXTERNALROUTINE LPPRED(%RECORDNAME PPRED,%STRING(4) MODE,NUMB)
     !
     !
     !
     !      REVISION  00
     !      --------
     !
     !      PURPOSE
     !      -------  THIS ROUTINE IS USED TO LIST THE PARAMETERS IN THE RECORD
     !      PPRED.
     !
     !      USE
     !      ---          LPPRED(PPRED,MODE,NUMB)
     !
     !      PPRED       IS THE RECORD TO BE LISTED AND HAS FORMAT PPREDFORM
     !      MODE        HAS THE VALUE 'LIST' OR 'PRED' ACCORDING TO WHETHER
     !                  LPPRED IS CALLED FORM THE PROGRAMS LISTPARM OR PREDICT.
     !      NUMB        IS THE DATA SET REFERENCE NUMBER ASSOCIATED WITH THE
     !                  RECORD PPRED.
     !
     !      ORIGINATOR
     !      ----------  S.H.C. HUGHES, DEPT. MEDICAL PHYSICS, R.I.E..
     !
     !
     !
     !
     %RECORDSPEC PPRED(PPREDFORM)
     %ROUTINESPEC LIST1(%RECORDNAME PPRED)
     %ROUTINESPEC LIST2(%RECORDNAME PPRED)
     %ROUTINESPEC LIST3(%RECORDNAME PPRED,%STRING(4) MODE)
     %ROUTINESPEC LIST4(%STRING(4) NUMB)
     !
     !
     !
     LIST1(PPRED)
     LIST2(PPRED)
     %RETURN %UNLESS 1<=PPRED_OPTION<=5
     LIST3(PPRED,MODE)
     LIST4(NUMB) %IF MODE="PRED"
     NEWLINE
     !
     !
     !
     %ROUTINE LIST1(%RECORDNAME PPRED)
          !
          !
          %RECORDSPEC PPRED(PPREDFORM)
          !
          !
```

```
      NEWLINES(3)
      PRINTSTRING("INPUT PARAMETERS") ; NEWLINE
      PRINTSTRING("----------------") ; NEWLINES(2)
      PRINTSTRING("OMEGA C REC") ; SPACES(29)
      PRINTFL(PPRED_OMEGACREC,3) ; PRINTSTRING(" CYCLES/CM") ; NEWLINE
      PRINTSTRING("L, LINE LENGTH IN SCAN") ; SPACES(18)
      PRINTFL(PPRED_CAPL,3) ; PRINTSTRING(" CMS.") ; NEWLINE
      PRINTSTRING("MAX. ALLOWED ALIASING ERROR IN M-DATA") ; SPACES(3)
      PRINTFL(100*PPRED_THETA,3) ; PRINTSTRING(" %") ; NEWLINE
      PRINTSTRING("M, MINIMUM NO. OF SAMPLES ALLOWED") ; NEWLINE
      PRINTSTRING("  IN FIRST PEAK OF S") ; SPACES(20)
      PRINTFL(PPRED_LITM,3) ; NEWLINE
      %END

  !
  !
  !
  %ROUTINE LIST2(%RECORDNAME PPRED)
      !
      !
      %RECORDSPEC PPRED(PPREDFORM)
      !
      !
      NEWLINES(3)
      PRINTSTRING("PREDICTED PARAMETERS") ; NEWLINE
      PRINTSTRING("--------------------") ; NEWLINES(2)
      PRINTSTRING("PHANTOM DIAMETER"); SPACES(24)
      PRINTFL(PPRED_CAPD,3) ; PRINTSTRING(" CMS.") ; NEWLINE
      PRINTSTRING("DETECTOR COLIMATOR DIAMETER") ; SPACES(13)
      PRINTFL(PPRED_DCOLDIAM,3) ; PRINTSTRING(" CMS.") ; NEWLINE
      PRINTSTRING("CELL SIZE") ; SPACES(31)
      PRINTFL(PPRED_LITDELR,3) ; PRINTSTRING(" CMS.") ; NEWLINE
      PRINTSTRING("M, NO. OF LINES IN SCAN") ; SPACES(17)
      WRITE(PPRED_CAPM,1) ; NEWLINE
      PRINTSTRING("L, NO. OF COUNTS/CELL IN AIR") ; SPACES(12)
      PRINTFL(PPRED_LITL,1) ; NEWLINE
      %END

  !
  !
  !
  %ROUTINE LIST3(%RECORDNAME PPRED,%STRING(4) MODE)
      !
      !
      %INTEGER J,D,M
      %LONGREAL TEMPRL
      %RECORDSPEC PPRED(PPREDFORM)
      !
      !
```

```
NEWLINES(3)
PRINTSTRING("SCANNER SETTINGS") ; NEWLINE
PRINTSTRING("----------------") ; NEWLINES(2)
PRINTSTRING("STEPPING MOTOR PULSE FREQUENCY") ; SPACES(10)
WRITE(PPRED_RNDSMPF,1) ; PRINTSTRING(" PPS") ; PRINTSTRING("  (")
PRINTFL(PPRED_SMPF,3) ; PRINTSTRING(" )") ; NEWLINE
PRINTSTRING("GATE PULSE SCALE FACTOR") ; SPACES(17)
WRITE(PPRED_RNDGPSF,1) ; PRINTSTRING("  (")
PRINTFL(PPRED_GPSF,3) ; PRINTSTRING(" )") ; NEWLINE
PRINTSTRING("COLLIMATOR DIAMETER : SOURCE") ; SPACES(12)
%IF PPRED_DCOLDIAM<=0.300 %THEN PRINTSTRING(" 5")  %C
     %ELSE PRINTSTRING(" 10") ; PRINTSTRING(" MM") ; NEWLINE
PRINTSTRING("                    : DETECTOR") ; SPACES(10)
WRITE(INT(10*PPRED_DCOLDIAM),1) ; PRINTSTRING(" MM") ; NEWLINE
PRINTSTRING("NO. OF BLANKLINES") ; SPACES(23)
WRITE(PPRED_BLANKLINE,1) ; NEWLINE
PRINTSTRING("DIAMETER OF PHANTOM") ; SPACES(21)
WRITE(INT(10*PPRED_CAPD),1) ; PRINTSTRING(" MM") ; NEWLINES(2)
PRINTSTRING("  GIVING:-") ; NEWLINES(2)
PRINTSTRING("NORMDIAM ( 2A/DELTA R )") ; SPACES(17)
PRINTFL(PPRED_RNDK,3) ; NEWLINE
PRINTSTRING("MAX. ALIASING ERROR IN M-DATA") ; SPACES(11)
PRINTFL(100*PPRED_RNDTHETA,3) ; PRINTSTRING(" %") ; NEWLINE
PRINTSTRING("M, MINIMUM NUMBER OF SAMPLES IN FIRST") ; NEWLINE
PRINTSTRING("  PEAK OF S") ; SPACES(29)
PRINTFL(PPRED_RNDLITM,3) ; NEWLINE
PRINTSTRING("L, NO. OF COUNTS/CELL IN AIR") ; SPACES(12)
PRINTFL(PPRED_RNDLITL,3) ; PRINTSTRING(" COUNTS/CELL") ;NEWLINE
%RETURN %IF MODE="LIST"
NEWLINES(3)
PRINTSTRING("ANGULAR SETTINGS ON MILLING TABLE") ; NEWLINES(2)
%CYCLE J=1,1,PPRED_CAPM
     SPACES(10)
     TEMPRL=180*(2-(J-1)/PPRED_CAPM)
     D=INT PT(TEMPRL)
     M=INT(60*TEMPRL)-60*D
     M=0 %AND D=D+1 %IF M=60
     WRITE(D,3)
     PRINTSTRING(" :")
     WRITE(M,2)
     PRINTSYMBOL('"')
     NEWLINE
     NEWLINE %IF J&B'11'=0
     %REPEAT
%END

!
!
!
```

```
%ROUTINE LIST4(%STRING(4) NUMB)
     !
     !
     NEWPAGE ; NEWLINES(10)
     SPACES(30) ; PRINTSTRING("DEPARTMENT OF MEDICAL PHYSICS") ; NEWLINE
     SPACES(30) ; PRINTSTRING("THE ROYAL INFIRMARY") ; NEWLINE
     SPACES(30) ; PRINTSTRING("EDINBURG, 3") ; NEWLINE
     SPACES(30) ; PRINTSTRING("229 2477 EXT 2600") ; NEWLINES(3)
     PRINTSTRING("PETER SCOTT") ; NEWLINE
     PRINTSTRING("JOB RECEPTION") ; NEWLINE
     PRINTSTRING("E. R. C. C. ") ; NEWLINES(3)
     SPACES(20) ; PRINTSTRING("BINARY DATA TRANSFER") ; NEWLINE
     SPACES(20) ; PRINTSTRING("--------------------") ; NEWLINES(2)
     PRINTSTRING("PLEASE TRANSFER THE FOLLOWING DATA INTO MY PROCESS:-")
     NEWLINES(2)
     PRINTSTRING("USER                       EPMP07") ; NEWLINE
     PRINTSTRING("TAPE                       ........") ; NEWLINE
     PRINTSTRING("FORMAT                     PDP 15") ; NEWLINE
     PRINTSTRING("BLOCKS (DECIMAL)           ...............") ; NEWLINE
     PRINTSTRING("FILE NAME                  DA".NUMB) ; NEWLINES(4)
     SPACES(30) ; PRINTSTRING("MANY THANKS") ; NEWLINES(4)
     SPACES(40) ; PRINTSTRING("S.H.C. HUGHES.") ; NEWLINE ; NEWPAGE
     %END

%END
```

```
%EXTERNALINTEGERFN CAPLAMDA(%INTEGER K,N)
     !
     !
     !
     !
     !      REVISION 00
     !      --------
     !
     !      PURPOSE
     !      -------  THIS FUNCTION RETURNS THE VALUE OF CAPITAL LAMBDA.
     !      IT ASSUMES THAT N IS AN INTEGRAL POWER OF 2.
     !
     !      COMMENTS
     !      --------
     !      1)   FOR FURTHER DETAILS AND  A DEFINITION
     !           OF CAPITAL LAMBDA SEE REPORT.
     !
     !      ORIGINATOR
     !      ----------  S.H.C. HUGHES, DEPT. MEDICAL PHYSICS, R.I.E..
     !
     !
     !
     %RESULT=K %IF K<=N>>1
     %RESULT=N-K
     %END

!
!
!
!
!
```

```
%EXTERNALINTEGERFN RADIX2(%INTEGER N)
      !
      !
      !     REVISION 00
      !     --------
      !
      !     PURPOSE
      !     -------   THIS ROUTINE WILL RETURN THE VALUE
      !                MIN(2**N" : 2**N">=N AND N,N" INTEGER)
      !
      !     USE
      !     ---        N2=RADIX2(60) ASSIGNS THE VALUE 64 TO N2
      !
      !     COMMENTS
      !     --------
      !     1)    IF N<=0 THE FN OUTPUTS THE MESSAGE:-
      !                "ARGUEMENT NON-POSITIVE IN ROUTINE CALL  RADIX2(N)"
      !           TO THE TT AND STOPS.
      !     2)    IF N>2**30=1 073 741 824 THE FN OUTPUTS THE MESSAGE:-
      !                "ARGUEMENT TOO LARGE IN ROUTINE CALL  RADIX2(N)"
      !           TO THE .TT AND STOPS.
      !
      !     ORIGINATOR
      !     ----------  S.H.C. HUGHES, DEPT. MEDICAL PHYSICS, R.I.E..
      !
      !
      !
      %INTEGER M,B
      %STRING(255) MESSAGE1,MESSAGE2
      !
      !
      MESSAGE1="ARGUEMENT NON-POSITIVE IN ROUTINE CALL   RADIX2(".CHAR(N).")"
      MESSAGE2="ARGUEMENT TOO LARGE IN ROUTINE CALL   RADIX2(".CHAR(N).")"
      !
      !
      M=X'80000000'
      CLOSE(MESSAGE1) %IF N&M=M
      B=BITS(N)
      %RESULT=N %IF B=1
      CLOSE(MESSAGE1) %IF B=0
      M=M>>1
      CLOSE(MESSAGE2) %IF N&M=M
      M=M>>1
      %WHILE M&N=0 %CYCLE ;  M=M>>1 ; %REPEAT
      %RESULT=M<<1
      %END

!
!
```

```
%EXTERNALROUTINE RLMATO(%LONGREALARRAYNAME X,%INTEGER START,FINISH)
    !
    !
    !
    !     REVISION 00
    !     --------
    !
    !     PURPOSE
    !     -------  THIS ROUTINE WILL ASSIGN THE VALUE ZERO TO THE
    !     ELEMENTS X(START) TO X(FINISH) OF THE LONGREAL ARRAY
    !     X.
    !
    !     ORIGINATOR
    !     ----------  S.H.C. HUGHES, DEPT. MEDICAL PHYSICS, R.I.E..
    !
    !
    !
    !
    !
    %INTEGER AD1,AD2,I
    !
    !
    AD1=ADDR(X(START))
    AD2=ADDR(X(FINISH))
    %CYCLE I=AD1,8,AD2
         LONGREAL(I)=0 ; %REPEAT
    %END
```

1. Purpose

C06ABF calculates the finite Fourier transform of $2^{(M1-1)}$ complex data values using the Cooley-Tukey algorithm. M1 is a positive integer not less than 3.

IMPORTANT: before using this routine, read the appropriate machine implementation document to check the interpretation of italicised terms and other implementation-dependent details.

2. Specification (FORTRAN IV)

```
      SUBROUTINE C06ABF(A,B,N1,INVERS,M1,C)
C     INTEGER N1,M1
C     LOGICAL INVERS
C     real A,B,C
C     DIMENSION A(N1),B(N1),C(M1)
```

3. Description

This routine uses the Cooley-Tukey algorithm to calculate the finite Fourier transform of $n = 2^{(M1-1)}$ complex data values. The arrays A and B must contain on entry the real and imaginary components of the data respectively and on exit they contain the real and imaginary components of the relevant transform.

If INVERS is .FALSE. on entry to the routine, the Fourier transform

$$x_j + iy_j = \frac{1}{\sqrt{n}} \times \sum_{k=1}^{n} (a_k + ib_k) \exp(i2\pi(j-1)(k-1)/n)$$

for $j = 1,2,\ldots,n$

is calculated, and if INVERS is .TRUE. on entry, the inverse Fourier tranform

$$a_k + ib_k = \frac{1}{\sqrt{n}} \times \sum_{j=0}^{n} (x_j + iy_j) \exp(-i2\pi(k-1)(j-1)/n)$$

for $k = 1,2,\ldots,n$

is calculated. In these formulae i denotes $\sqrt{-1}$.

C06ABF

4. References

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5. Parameters

A - *real* array of DIMENSION at least (N1).
Before entry, A must contain the real components of the data and on exit, it contains the real components of the relevant transform
i.e. A(j) must contain either a_j or x_j before entry and on exit it contains x_j or a_j respectively, depending on the value of INVERS on entry.

B - *real* array of DIMENSION at least (N1).
Before entry, B must contain the imaginary components of the data and on exit, it contains the imaginary components of the relevant transform
i.e. B(j) must contain either b_j or y_j before entry and on exit it contains y_j or b_j respectively, depending on the value of INVERS on entry.

N1 - INTEGER.
On entry, N1 must specify the minimum dimensions of the arrays A and B where $N1 \geq 2^{(M1-1)}$. Unchanged on exit.

INVERS - LOGICAL.
On entry, INVERS specifies whether the Fourier transform or the inverse Fourier transform is to be calculated. If INVERS is .FALSE. on entry, the Fourier transform is calculated, and if it is .TRUE. on entry, the inverse Fourier transform is calculated. Unchanged on exit.

M1 - INTEGER.
On entry, M1 specifies M1, such that the number of data values to be used in the transformation, n, is given by $n=2^{(M1-1)}$.

5. Parameters (contd)

C - *real* array of DIMENSION at least (M1).
The array C is used as a work array by the routine and its contents are not specified on exit.

6. Error Indicators None.

7. Auxiliary Routines

This routine calls the NAG Library routines C06AAY, C06ABZ and X01AAF.

8. Timing

The computation time is dependent on the number of data values and is proportional to $M \times 2^M$ where $M = M1-1$.

9. Storage

There are no internally declared arrays.

10. Accuracy

Basic precision arithmetic is used throughout. The accuracy will depend on the range of the data values specified.

11. Further Comments

The routine will fail if the number of data values specified is greater than the largest integer representable on a computer. This value may be calculated using X02BBF.

12. Keywords

Complex Finite Fourier Transform.
Fast Fourier Transform.
F.F.T.

Appendix 3·6: Details of error terms in $\hat{f}$

·1 Regularisation error

By definition $S = \dfrac{|R|\lambda_c}{|R|^5 + \lambda_c}$ and $s = \mathcal{F}_1^{-1}S$ (§2·541).

The functions $\Sigma(R)$ and $\sigma(r)$ are now defined by

$$\Sigma = \frac{\lambda_c}{|R|^5 + \lambda_c} \quad \text{and } \sigma = \mathcal{F}_1^{-1}\Sigma.$$

With these definitions and the results of §1·1·462 and §1·1·444, then

$$\begin{aligned}
\mathcal{B}(s * p) - f &= \mathcal{B}\mathcal{F}_1^{-1}|R|\mathcal{F}_2(\sigma * p) - f \\
&= \mathcal{R}^{-1}(\sigma * p) - f \\
&= \mathcal{F}_2^{-1}\mathcal{F}_1(\sigma * p) - f \\
&= \mathcal{F}_2^{-1}(\Sigma F - F) \\
&= \mathcal{F}_2^{-1}\left(\frac{-|R|^5}{|R|^5 + \lambda_c} \quad F\right) \\
&= \mathcal{F}_2^{-1}\left(\frac{-|R|^5}{|R|^5 + \lambda_c}\right) * f
\end{aligned}$$

·2 Collimater and regularisation error

From §5 the data processing when no deconvolution is performed is

$$\begin{aligned}
\hat{\underline{g}} &= \ell^{-1}\underline{m} \\
\hat{\underline{q}} &= \hat{\underline{g}} \\
\hat{\underline{p}} &= -\ln \hat{\underline{q}}
\end{aligned}$$

hence

$$\begin{aligned}
E\{\hat{\underline{g}}\} &= \ell^{-1}E\{\underline{m}\} = \ell^{-1}\underline{\mu} = \underline{g} \\
E\{\hat{\underline{q}}\} &= E\{\hat{\underline{g}}\} = \underline{g} \\
E\{\hat{\underline{p}}\} &\simeq -\ln E\{\hat{\underline{q}}\} = -\ln \underline{g}
\end{aligned}$$

where $\ln \underline{g}$ is a vector defined by $(\ln \underline{g})_k = \ln(\underline{g})_k$ and the last approximation is true only if $\mathrm{var}(q_j) \ll E\{q_j\}$ $\forall j$.

It follows immediately that

$$
\begin{aligned}
E\{\hat{\underline{p}} - \underline{p}\} &= E\{\hat{\underline{p}}\} - \underline{p} \\
&= - \ln \underline{g} - \underline{p} \\
&= - \ln \underline{g} - (- \ln \underline{q}) \\
&= \ln(\underline{q}/\underline{g})
\end{aligned}
$$

(where $\ln(\underline{q}/\underline{g})$ is a vector defined by $(\ln(\underline{q}/\underline{g}))_k = \ln(q_k/g_k)$) thus $s * E\{\hat{\underline{p}} - \underline{p}\}$ is taken to be the continuous version of $\underline{\underline{s}}\ \ln(\underline{q}/\underline{g})$ and $\mathscr{B}(s * E\{\hat{\underline{p}} - \underline{p}\})$ is identified with $\mathscr{B}(s * \ln(q/g))$.

·3 Statistical error

By definition $E\{\epsilon\} = 0$ thus

$$\mathrm{var}(\mathscr{B}(s * \epsilon)) = E\ \{\mathscr{B}(s * \epsilon)^2\}\ .$$

But by definition (§1·1·461)

$$(\mathscr{B}t)(x) = \frac{1}{2} \int_{S^{n-1}} d\phi \quad t(\langle x,\phi\rangle,\ \phi\ )$$

so that

$$(\mathscr{B}(s * \epsilon\ ))(x) = \frac{1}{2} \int_{S^{n-1}} d\phi \int_{\mathbb{R}} dr\ s(\zeta - r)\ \epsilon(r,\phi)$$

where $\zeta = \langle x,\phi\rangle$, and the fact that $s(r)$ is independent of ϕ has been incorporated. (Note that the original n-dimensional notation of appendix 1·1·4 is being used and not the 2-D notation of chapters 2 and 3 so that n denotes the dimension of the space (and not the power of $|R|$ in $S(R)$) and ϕ is a unit n-vector not an angle.) In addition there is the result

$$
\begin{aligned}
&E\{\varepsilon(r_1,\phi_1)\quad \varepsilon(r_2,\phi_2)\} \\
&\quad = \mathrm{var}\left(\epsilon(r_1,\phi_1)\right)\quad \{\delta(r_1 - r_2)\delta(\phi_1 - \phi_2) + \delta(r_1 + r_2)\delta(\phi_1 + \phi_2)\}.
\end{aligned}
$$

(The second term in this expression arises as follows: by definition $\mu(r,\phi) = \mu(-r,-\phi)$, thus when measuring $m(r,\phi)$ samples are taken for $r \in [-\infty,\infty]$ and for ϕ ranging over half of S^{n-1} and there is only one sample for both the points $(r)(\phi)$ and $(-r)(-\phi)$.)

Combining these definitions and results:-

$$E\{(s * \epsilon)^2\} =$$

$$E\left\{ \frac{1}{2}\int_{S^{n-1}} d\phi_1 \frac{1}{2}\int_{S^{n-1}} d\phi_2 \int_{\mathbb{R}} dr_1 \int_{\mathbb{R}} dr_2 \; s(\zeta_1 - r_1)s(\zeta_2 - r_2)\epsilon(r_1,\phi_1)\;(r_2,\phi_2)\right\}$$

$$= \int_{S^{n-1}} d\phi_1 \frac{1}{2}\int_{S^{n-1}} d\phi_2 \int_{\mathbb{R}} dr_1 \int_{\mathbb{R}} dr_2 \; s(\zeta_1 - r_1)s(\zeta_2 - r_2)\mathrm{var}(r_1,\phi_1)\begin{Bmatrix}\delta(r_1-r_2)\delta(\phi_1-\phi_2) \\ +\delta(r_1+r_2)\delta(\phi_1+\phi_2\end{Bmatrix}$$

$$= \int_{S^{n-1}} d\phi_1 \int_{\mathbb{R}} dr_1 \; \mathrm{var}(\epsilon(r_1,\phi_1))\; s(\zeta_1 - r_1)^2$$

$$= \mathcal{B}(s^2 * \mathrm{var}(\epsilon))$$

where the fact that $s(r) = s(-r)$ has been used during the course of the tidying up.

Further from the definition of ϵ and from §5

$$\mathrm{var}(\underline{\epsilon}) = \mathrm{var}(\hat{\underline{p}}) = (\ell \underline{g})^{-1}$$

where $\underline{g}^{-1}$ is the vector $(\underline{g}^{-1})_k = ((\underline{g})_k)^{-1}$. Thus $E\{\mathcal{B}(s * \epsilon)^2\}$ may be identified with $\mathcal{B}(s^2 * (\ell g)^{-1})$.

Addendum

The following is a list of typing errors found after submission of the thesis.

page	line	correction
iii	3	delete 'as'
9	24	'Those' becomes 'These'
20	14	'there' becomes 'these'
20	28	insert '§'
21	2	'amounts' becomes 'amount'
23	19	'inventing' becomes 'inverting'
24	6	'chapter 1' becomes 'Preface ·1 and ·2'
24	23	'F 4' becomes 'F·4'
29	19	'd ' becomes 'dα'
31	25	'invented' becomes 'inverted'
33	9	'E( )' becomes 'E{ }' twice
49	21	')$\mathcal{F}_1\hat{p}$' becomes '$\mathcal{F}_1\hat{p}$)'
53	16	'λ_c R ' becomes 'λ_c $\|R\|$ '
56	12	'points' becomes 'points,'
63	4	after equation insert equation number '·2'
77	9	'invertion' becomes 'inversion'
94	24	'§5' becomes '§5·45'
98	9	'$\omega_{c\ col}$ $\omega_{c\ fil}$' becomes '$\omega_{c\ col} \gg \omega_{c\ fil}$'
109	8	'impurity' becomes 'impunity'
110	4	'δ' becomes 'ω'
118	7	'derived' becomes 'desired'
120	5	'f,' becomes 'f_o,'

120	17	'ΔC_1' becomes 'ΔC_{t1}'
120	19	'4' becomes '3'
120	21	'$f_0 \Delta p$ x' becomes '$f_o p\Delta x$'
120	25	'4 d^2' becomes '4vd^2'
138	31	'55 nA' becomes '55 pA'
140	27	'800 Overall' becomes '800 < Overall'
166	27	'rows' becomes 'columns'
178	5	'$\underline{\ell}(\phi)$' becomes '$\underline{\ell}(\phi_j)$'
179	24	'$\mathcal{L}$,' becomes '$\mathcal{L}^1$,'
181	11	'correctedly' becomes 'correctly'
240	23	'$\omega_{c\ fil}$' becomes '$\omega_{c\ ang}$'
271	11	'$\int_{\infty}$' becomes '$\int_{-\infty}$'
272	12	' 15' becomes '·15'
272	13	' f_j ' becomes '$\{f_j\}$' and ' F_k ' becomes '$\{F_k\}$'
284	3	'transform f' becomes 'transform $\mathcal{R}$f'
284	9	' f defined in' becomes '$\mathcal{R}$f defined on'
286	2	'$\| \ f\|^2$' becomes '$\|\mathcal{R}f\|^2$'
287	3	'normal' becomes 'normed'
287	13	'Radon' becomes 'Rudin'
288	2	'of .' becomes 'of $\mathcal{R}$.'
289	3	'} ' becomes '})'
289	14	'$\| \ X\|$' becomes '$\|\Delta X\|$'
290	9	'$\|\|\mathcal{F}f\|\|$' becomes '$\|\|\mathcal{F}_1 f\|\|$'
290	17	'$\underline{\mathcal{P}}$F' becomes '$\mathcal{P}$F'
291	11	'5 4' becomes '5·4'
291	14	' $_n(C(B^n))$' becomes ' $\mathcal{F}_n C(B^n))$'
302	4	'S' ' becomes 'S^1 '
308	1	'2·2·2' becomes '2·1·2'
309	1	'2·2·3' becomes '2·1·3'
309	6	'$\underline{\underline{W}}$ $\underline{h}$* ' becomes '$\underline{\underline{W}}$ $\underline{h}$* and H'

310	1	'2•2•4' becomes '2•1•4'
311	1	'2•2•5' becomes '2•1•5'
321	4	'diag($\underline{\underline{\Lambda}}$)' becomes 'diag($\underline{\Lambda}$)'
328	5	delete 'Accordingly $\underline{s}''$ is considered an accurate estimate of s and $\underline{s}'$. '
368	15	') r_2' becomes ') Δr_2' twice
377	20	'$\rho/4$' becomes '$\pi/4$'
398	14	'§5' becomes '§5•4'
414	4	'sumbol' becomes 'symbol'
415	4	'§3•442' becomes '§3•4•42'
421	18	'n = a(b)c' becomes 'x = a(b)c'

Addendum

This space is reserved for future corrections.

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Index of Notation

The following general rules may help the reader:-

1) Within the main text (i.e. excluding appendices) each important quantity usually has a sumbol of its own. In those situations where the same symbol has two meanings every attempt has been made to make the secondary use explicit in the text.

2) Without exception, when a C.F.T. or D.F.T. pair are being considered the real and Fourier space functions (vectors) are denoted by lower and upper case symbols respectively. As far as possible this is applied to coordinates as well as functions.

3) Within the main text vectors and matrices are denoted by single and double underlining.

4) Operators on function spaces, e.g. Fourier transforms, are denoted by upper case script Roman letters.

A list of more important symbols is given below. In addition the reader should become thoroughly familiar with §1·1·1 and should bear in mind §1·1·42.

Roman script

$\mathcal{B}$	§1·322	Back projection operator (see also §1·1·461 and §1·1·47)
$\mathcal{C}$	§1·32312	Correction operator
$\mathbb{C}$		set of complex numbers
$\mathcal{F}_n$	§1·321	n-dimensional Fourier transform
$\mathcal{L}^p$		set of p-th power Lebesgue integrable functions
$\mathbb{N}$		natural numbers
$\mathcal{P}$	§1·1·4431	a mapping from part of $\mathcal{L}^2(\mathbb{R}^n)$ to $\mathcal{L}^2(\mathbb{R} \times S^{n-1})$
$\mathcal{R}$	§1·3	Radon transform (see also §1.1.43)
$\mathbb{R}^n$	§1·321	n-dimensional Euclidean space, $\mathbb{R} = \mathbb{R}^1$

Upper case Roman

A	§2·5421	value of an integral
$A(\ell)$	§3·442	(also written $A(\ell,\phi, \Delta r_2,\Delta r',N_3,N_4)$)
B^n	§2·5	ball of radius $D/2$ in $\mathbb{R}^n$
B^n	§1·1·42	unit ball in $\mathbb{R}^n$
C	§4·21	number of counts/cell in air
C_t		
C_{t1}	§4·21	number of counts/cell through tissue
C_{t2}		
C'	§4·21	detector count rate
C'_{max}	§4·21	maximum count rate of which detector is capable
$\underline{\underline{C}}$	§2·32	D.F.T. of $\underline{\underline{c}}$
D	§1·31	diameter of patient section
E	§2·11	detector efficiency
$E\{\ \}$	§2·12	statistics : expected value
F	§1·321	C.F.T. of f
G	§2·31	C.F.T. of g
H	§2·31	C.F.T. of h
I	§1·2	γ-ray intensity
$J_n(x)$		n-th order Bessel function
L	§2·32	Lagrangian
$L(\alpha,\theta)$	§1·2	line of integration
$\underline{\underline{L}}$	§2·372	D.F.T. of $\underline{\underline{\ell}}$
L	§3·424	traverse length
M	§1·41	number of projections (see also §2·13)
$\underline{\underline{M}}(\phi)$	§2·12	covariance matrix for measured data
$N(\underline{\mu},\underline{\underline{V}})$		Normal distribution with mean $\underline{\mu}$ and covariance $\underline{\underline{V}}$
N	§2·11	number of observations per traverse
N_1	§2·21	number of data values used to estimate $\hat{\ell}$
N_2	§2·321	number of coordinates in $\underline{g}$
N_3	§2·531	number of coordinates in $\underline{p}$

N_4	§2·5422	number of samples in peak of $S(R)$
N_4	§3·4311	phantom is an $N_4 \times N_4$ matrix
P	§1·321	C.F.T. of p
$P_s^{(\alpha,\beta)}$	§1·42	Jacobi polynomial
Q	§2·31	C.F.T. of q
R	§1·321	first polar coordinate e.g. (R,Φ)
R^n_{n+2s}	§1·42	Zernike polynomial
S	§2·541	C.F.T. of s
$S(f)$	§2·51	smoothness measure
S^{n-1}	§1·1·42	unit sphere in $\mathbb{R}^n$
$U_n(x)$	§1·42	Chebyshev polynomial
$\underline{\underline{V}}(\underline{x})$	§2·12	covariance matrix of vector $\underline{x}$
$\underline{\underline{W}}$	§1·1·15	D.F.T. operator

Lower case Roman

a	§2·11	collimator or beam radius
c	§3·4·41	$\cos\phi$
$\underline{\underline{c}}$	§2·32	2nd difference operator
$\mathrm{cov}(x,y)$		covariance of x and y
d	§4·21	source-detector distance
$d(f_1,f_2)$	§2·51	metric on $\mathcal{L}^2$
$\mathrm{dev}(j,k,\ell,\phi)$	§3·4·44	
$\mathrm{diag}(\underline{x})$		matrix such that $(\mathrm{diag}(\underline{x}))_{jk} = (\underline{x})_j\delta_{jk}$
e		base of natural log
e^2	§2·32	sum of squared errors
e^2	§2·521	sum of squared errors
e^2_c	§2·532	sum of squared errors associated with functions defined on a continuum
e^2_d	§2·532	sum of squared errors associated with a vector
$f(r,\phi)$	§1·2	linear attenuation coefficient in section (polar coordinates)
$f(r,\phi,z)$	§2·11	linear attenuation coefficient in section (polar coordinates)
$f(x,y)$	§3·4311	linear attenuation coefficient in phantom (Cartesian coordinates)

f_{jk}	§3·4311	linear attentuation coefficient in j,k th cell of phantom
f_0	§4·21	constant linear attenuation coefficient
g	§2·11	$g = h * q$
$\underline{g}$	§2·221	discrete version of $g(r)$
$h(r)$	§2·13	$h_1 * h_2$
$h_1(r)$	§2·11	collimator filter function (see also §3·2)
$h_2(r)$	§3·2	sampling filter function
i		$\sqrt{-1}$
j		integer suffix
k		integer suffix
k	§3·2	normalised collimator diameter $2a/\Delta r$
ℓ		integer suffix
ℓ	§2·11	number of counts/cell in air
$\underline{\underline{\ell}}$	§2·371	inversion matrix for $\underline{g} = \underline{\underline{h}}^{*}\underline{q}$
ln		$\log_e$
m		integer suffix
$\underline{m}$, $\underline{m}(\phi)$	§2·12	observed data for traverse at angle ϕ
m	§2·5431	number of samples in half cycle of $s(r)$
$\min\{x\}$		minimum element of set $\{x\}$, also written $\min\{x_1,x_2,x_3,x_4\}$
n		integer suffix
n	§2·11	number of photons/sec along beam
n	§2·531	parameter in s filter
n	§1·1·4	dimension of Euclidean space
n	§4·21	number of photons/sec radiated by source
p	§1·2	line integral of f (see also §2·11)
$p^{*}(r/\Delta r_2)$	§3·4·432	also written $p^{*}(r/\Delta r_2, \Delta r_2, \phi)$
p	§4·21	percentage increase in attenuation coefficient
q	§2·11	$q = \exp(-p)$
r	§1·31	first polar coordinate e.g. (r,ϕ)

r	§4·38	unspecified value of resistor
r_{max}	§3·4·41	
r_{mid}	§3·4·41	
r_{min}	§3·4·41	
s	§3·4·41	$\sin \phi$
s	§1·42	integer suffix
$s(r)$	§2·541	$\underline{p}$ data filter function
s_1	§1·322	filter function
s_2	§1·41	filter function
s_3	§1·41	filter function
$\mathrm{sinc}(x)$		$\sin(\pi x)/\pi x$
t		denotes general function
t	§2·11	time
$\mathrm{tr}(\underline{\underline{M}})$		trace of matrix $\underline{\underline{M}}$
v	§2·11	traverse velocity
$\underline{v}(r)$	§2·531	interpolation vector
$\underline{v}_1(r)$	§2·531	step function interpolation vector
$\underline{v}_2(r)$	§2·531	linear interpolation vector
$\mathrm{var}(x)$		variance of x
x		first Cartesian coordinate
x'	§4·21	path length through section
y		second Cartesian coordinate
z	§2·11	third polar coordinate

Upper case Greek

Δr	§2·11	radial sample spacing
$\Delta r'$	§3·4312	sample spacing used for prediction
Δr_2	§3·4311	cell size in phantom
ΔR		sample spacing in Fourier space. N.B. This is variable e.g. $\Delta R = (N_2 \Delta r)^{-1}$ for $\underline{g}$ $\Delta R = (N_3 \Delta r)^{-1}$ for $\underline{p}$

Δx	§4·21	size of cell in section
$\Delta\phi$	§2·13	angular sample spacing
$\Lambda(x)$	§1·41	$\Lambda(x) = \begin{cases} 1 - \lvert x\rvert & \lvert x\rvert < 1 \\ 0 & \lvert x\rvert > 1 \end{cases}$
$\underline{\underline{\Lambda}}$	§2·531	ramp matrix
Φ	§1·321	second polar coordinate e.g. (R,Φ)

Lower case Greek

α	§1·2	first polar coordinate e.g. (α,θ)
γ	§2·32	inverse of Lagrange multiplier
γ	§4·38	shortening ratio of pulse shaping circuit
δ		small quantity
δ_{jk}		Kroneker delta
$\delta(r)$		delta function
ε		small quantity
ε		error
ε		set theory : contained in
θ	§1·2	second polar coordinate e.g. (α,θ)
θ	§2·5421	aliasing error in $S(R)$
θ	§2·222	fractional error in $\hat{\ell}$
θ_1	§3·2	aliasing error in $h(r)$
θ_2	§2·5431	fractional aliasing error in $\underline{s}$
θ_j	§2·531	angular position of j-th traverse
λ	§2·32	Lagrange multiplier
λ_c	§2·521	Lagrange multiplier associated with functions defined on a continuum
λ_d	§2·531	Lagrange multiplier associated with vector
$\lambda_{d\,min}$	§2·5431	minimum value of λ_d
$\lambda_{d\,max}$	§2·5431	maximum value of λ_d
λ_{ns}	§1·42	s-th zero of $J_n(x)$
μ	§2·11	expected number of counts at one point of traverse

π		as usual
ρ	§1·42	first polar coordinate
τ	§4·38	decay time constant of scintillation pulse
ϕ	§1·321	second polar coordinate e.g. (r,ϕ)
ϕ_j	§2·13	angular position of j-th traverse
χ^2	§2·24	χ^2 distribution
ψ_j, ψ_{ns}	§1·42	basis functions
ω_c	§1·41	cut-off frequency
$\omega_{c\ ang}$	§3·313	angular cut-off frequency
$\omega_{c\ col}$	§3·311	collimator cut-off frequency
$\omega_{c\ fil}$	§3·312	s filter cut-off frequency
$\omega_{c\ rec}$	§3·41	reconstruction cut-off frequency

Miscellaneous

$\underline{\cdot}$		denotes a vector
$(\underline{\cdot})_k$		denotes the k-th coordinate of the vector $\underline{\cdot}$. N.B. note distinction between $(\underline{x})_2$, $\underline{x}_2$ and x_2 which are the second component of $\underline{x}$, a vector called "x two" and a scalar called "x two" (which may or may not be the second component of $\underline{x}$) respectively.
$\underline{\underline{\cdot}}$		denotes a matrix
$(\underline{\underline{\cdot}})_{jk}$, $(\underline{\underline{\cdot}})_{j,k}$		denotes j k th element of $\underline{\underline{\cdot}}$
$.'$	§1·322	used to distinguish between related quantities e.g. x, x', x".
$.'$	§1·1·16	denotes transpose of vector or matrix
$\hat{\cdot}$	§1·32311	denotes an estimate e.g. $\hat{f}$ is an estimate of f
$.^*$		used to distinguish between related quantities e.g. h and h^*
$.^T$		complex conjugate transpose

$\bar{\cdot}$		closure of a set
$\bar{\cdot}$		complex conjugate
$\Leftrightarrow$		logical equivalence
$\left.\begin{matrix}\Rightarrow\\ \Leftarrow\end{matrix}\right\}$		implication
$\forall$		for all
$\ni$		such that
$\exists$		there exists
∇		del operator
$*_1$	§1·321	one dimensional convolution usually with respect to first polar variable
$*_2$	§1·41	two dimensional convolution
$\langle \underline{x}_1, \underline{x}_2 \rangle$		inner product on $\mathbb{R}^n$
$\lVert f \rVert$	§1·1·42	norm of f
$\{\ \}$		set or sequence
$[a,b]$		closed interval
$j\|N$		j modulo N (N.B. value between 0 and $N-1$ not 1 and N)
$n = a(b)c$		$\{x \mid x = a + \ell b,\ \ell \geq 0,\ \ell \in \mathbb{N} \text{ and } x \leq c\}$
$\Pi(x)$		$\Pi(x) = \begin{cases} 1 & \|x\| < 1/2 \\ 0 & \|x\| > 1/2 \end{cases}$
$\rightarrow$		tends to e.g. $x^{-1} \rightarrow 0$ as $x \rightarrow \infty$
$A \times B$		Cartesian product of sets A and B
$[x]$		$\sup(\{y\|y \leq x\} \cap \mathbb{N})$ i.e. integer part of x
$]x[$		fractional part of x, i.e. $x - [x]$

Abbreviations

C.F.T.	continuous Fourier transform
D.F.T.	discrete Fourier transform
E	equation (see preface ·5)
F	figure (see preface ·5)
F.F.T.	fast Fourier transform
F.T.	Fourier transform
§	section (see preface ·5)